An approach to the height datum unification problem based on a fixed mixed boundary value problem

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Abstract: In this paper a method for solving the problem of height datum unification is presented. This is essentially a problem for the determination of the potential differences among different height datums. The local height datums vary mainly due to different ways of their definition, methods of realizations and the fact that they are based on local data. The main approaches for determining potential differences are outlined and compared, taking into account the recent developments of the theory of geodetic boundary value problems (BVPs). This allowed us to select the fixed mixed BVP as the most suitable type for the estimation of the quasigeoid which has the advantage that is independent of any local height datums and it can be regarded as a global height datum. The basic method of datum unification relies on the comparison of the potential differences of each local height datum with the so-determined global height datum (i.e. the quasigeoid).

1. Introduction

One of the present day challenges of geodesy is the unification of all local and regional height datums into one consistent height datum. The practical problem is to realize a global reference surface supporting geometric (e.g. from GPS) and physical heights (e.g. from levelling, sea level observations) and to integrate the existing local height systems into one global system that is compatible with international standards and enables cost-saving implementation of modern (satellite, terrestrial, airborne and shipborne) geodetic techniques with accuracies ranging from 10^{-8} to 10^{-9} globally.

For a long time, Mean Sea Level (MSL) has been regarded as the reference surface for heights. MSL expresses a state of gravitational equilibrium and is generally determined as the average height of the ocean's surface measured by long-term sea level observations in one or several tide gauges (Zhang et al., 2009). However MSL is not an equipotential surface of the Earth's gravitational field, because in reality, due to currents, air pressure, temperature and salinity variations, etc., this does not occur, not even as a long term average. Therefore, different height datums refer to different equipotential surfaces, and consequently there exist various offsets between different local height datums with respect to the chosen 'reference surface'. In addition, the MSL and the geoid are not the same. The geoid describes the irregular shape of the earth and is the true zero surface for measuring elevations, since it is an equipotential surface of the Earth's gravity field that approximates the global MSL in the least squares sense. In a state of rest or absence of external non-gravitational forces, MSL would coincide with this geoid surface. The deviation between MSL and the geoid can vary globally in as much as ±2 m and is often referred to as stationary Sea Surface Topography or SST for short (Ardalan and Safari, 2005). In some oceanic regions, like the equatorial areas, the assumptions about a stationary SST do not hold, and consequently the marine geoid in these areas has to computed separately (in patches) for different zones that cannot be directly connected. Therefore, what is defined as 'zero elevation' in one region is not the same zero elevation defined in another region, which is why locally defined height datums differ from each other and need to be inter-connected.

Ideally, a global height datum conforming to the modern accuracy standards is required in order to serve many of the tasks of geodesy today, such as: to study SST at different tide gauges, construct regional or global geospatial information systems, monitor global climate changes by measuring long-term MSL variations, reduction in polar ice-cap volumes, post-glacial rebound and land subsidence studies, compute reliable estimates of ocean currents, etc. All of these applications require a global view of the Earth with measurements not only on land, but over the oceans as well (Fotopoulos, 2003).

In this paper, we approach this height datum unification problem through the determination of potential difference between two (or more) local height datums based on the solution of the linear fixed mixed (altimetry-gravimetry) boundary value problem. This allows obtaining the quasigeoid (instead of the geoid) which, although is not a level surface (in continental areas), and therefore, has no physical meaning, is a computationally convenient reference surface that is independent of any local height datums and can be regarded as a global height datum.

2. Approaches for determining potential differences

In general, there are three main approaches that can be followed in order to determine potential differences: (i) the classical (ii) the oceanographic and (iii) the Geodetic Boundary Value Problem (GBVP) approach.

In the classical approach, potential differences can be determined as the result of spirit levelling combined with gravity measurements. This involves a process that is repeated in a leap-frog fashion to produce elevation differences between established bench marks that comprise the vertical control network in the area of interest. When considering an arbitrary point P_o at sea level and another point P connected to P_o, the potential difference between P and P_o can be determined as

$$C = C(P, P_o) = W(P_o) - W(P) = W_O - W_P = -\int_{Po}^{P} dW = -\int_{Po}^{P} g\, dn \quad (1)$$

where C is known as the geopotential number of P that denotes the difference between the Earth's actual potential $W_O=W(P_o)$ at the geoid and the actual potential $W_P=W(P)$ of the surface on which the point P resides; g and dn denote respectively the average value of actual gravity and the elevation increment between successive benchmarks. Being a difference between geopotential values, the geopotential number C is independent of the levelling route along which the levelling is run in order to relate the height of point P to the sea level (at point P_o). Geopotential numbers make possible to estimate the orthometric H and the normal H^* height of a point, in the adopted local height datum, by using the following simple relations

$$H = C / \bar{g} \quad (2)$$

$$H^* = C / \bar{\gamma} \quad (3)$$

where $\bar{g}$ is the mean gravity along the actual plumb line from point P_o on the geoid up to point P on the surface of the Earth and $\bar{\gamma}$ is the mean value of the normal gravity from the surface of the Earth down to the quasigeoid along the normal plumb line. True orthometric heights are never achieved since their computation requires knowledge of or assumptions about the behavior of g inside the Earth (e.g. due to variations of the crustal density) where the mass distribution is unknown, and because it is also impossible to measure actual gravity along the plumb line, inside the Earth's topography. Normal heights on the other hand, they do not have these problems. Normal gravity can be calculated at any point without any hypotheses, as it is a simple analytical function of position depending only on the defining parameters of the reference level ellipsoid, which generates the normal gravity field. Hence, the normal height of a point P on the physical surface of the Earth can be interpreted as the height above the quasigeoid or alternatively above the telluroid, as it will be explained later on. The quasigeoid is identical with the geoid over the oceans and is very close to the geoid anywhere else. Its main advantage is that it can be computed rigorously without the necessity to make any hypotheses about the density distribution of the topographic masses, which accompanies the task of geoid determination (Heiskanen and Moritz, 1967). Once the quasigeoid is determined, it can be transformed into a geoid (if it is so desired) by introducing the desired hypothesis about the density of the topographic masses.

In spite of their obvious shortcomings (e.g. being time consuming, costly, laborious and suffering from problems of accumulation of the errors), this type of definition of height datums might be sufficient for applications of local or regional scale but would cause significant problems, as soon as connection of the height networks of different countries or continents separated by very wide areas and/or by oceans and unification of height datums in global scale are concerned (Colombo, 1980; Rummel and Teunissen, 1988; and Xu and Rummel, 1991).

In the oceanographic approach, geostrophic and steric sea level variation procedures are applied to the problem of determining the potential difference between two (or more) points across widely separated oceanic areas. These potential differences on the sea surface can be estimated from analyses of historical ocean subsurface temperature and salinity observations and/or inferred, for instance, from satellite altimetry merged mean sea anomalies (since 1993) and GRACE gravimetry (more recently) or from tide gauge data (over the past decades). This type of height datum unification is based on the presumption that the ocean acts as a huge level that can connect the zero points of the height datums realized by the reference tide gauges. However, the accuracy of ocean levelling is relatively low, mainly because the phenomena involved are very complex and difficult to model, but also due to many practical drawbacks, such as: the sparseness of ocean data (salinity, temperature, velocities of ocean currents), the time variability of the ocean, the inadequate knowledge of the ocean mass changes (e.g. due to change in atmospheric water, land hydrology and land ice mass), the non-validity of the geostrophic assumption about ocean currents, the poor reliability of satellite radar altimetry close to the coast, and the lack of precise tide models (Rummel and Ilk, 2009; Zhang et al., 2009; and Ardalan and Safari, 2005).

Under the framework of GBVPs, the potential difference between two (or more) areas can also be applied for height datum unification by introducing the local height datum discrepancies directly into the GBVPs (e.g., Rummel and Teunissen, 1988; Lehmann, 2000; and Ardalan et al., 2010). Using gravity measurements and levelling, only potential differences can be obtained, whereas the absolute value of the geopotential cannot be obtained at any point with acceptable accuracy. Consequently, the boundary values of the geopotential must be assumed to be known except for one additive constant that must be determined by imposing a suitable additional constraint (Sacerdote and Sansò, 2003). However, these methods require the use of local heights, e.g. in order to calculate the gravity anomalies. Furthermore, they can be affected by inconsistencies in the gravity data coming from different sources, which may have different datums or processed by inconsistent methods. In these cases, such uncertainties can be misinterpreted as height datum discrepancies.

This GBVP approach is the most recent one, and since it represents the starting point of our present work, it is discussed briefly in the following sections of the paper, in an effort to highlight, what is the most suitable GBVP formulation for determining the sought potential differences among various height datums (i.e. local vis-à-vis global, local indirectly to other local), by estimating the height datum discrepancies as follow up step after the BVP solution.

3. Formulations of geodetic boundary value problems

GBVPs represent a well-established basis of the analysis of terrestrial and satellite-based geodetic measurements for inference of the gravity field of the Earth as well as the quasigeoid or the geoid. The treatment of boundary value problems (BVPs) has always been used in geodesy as a suitable framework for determining the Earth's disturbing potential T (the difference between W and U, the actual and the normal potential respectively, where both quantities are referred to the same point). The classical theory of the GBVPs originated initially from the works of G.G. Stokes (ca. 1849) and M.S. Molodensky (ca. 1945), and was followed, in recent years, by more complicated formulations attempting to approach the real world conditions more realistically (i.e. be closer to the measurements), while also dealing with the issues of well-posedness (i.e. existence, uniqueness, and continuous dependence of the solution on boundary data). Depending on the type of data, several BVPs can be defined. However, after linearization around a suitable approximate solution all problems are special cases of a problem for the Laplace equation in the Earth's exterior. The boundary conditions associated with the GBVPs, in general, has the form of the so-called *fundamental equation of physical geodesy* (Heiskanen and Moritz, 1967)

$$\frac{\partial T}{\partial h} - \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T = -\Delta g, \tag{4}$$

where T is the disturbing potential, γ is the normal gravity, h is the geometric (ellipsoidal) height, ∂h denotes the partial derivative with respect to the direction of the normal plumb line and Δg denotes the gravity anomalies defined on the boundary surface being considered. This boundary is not the Earth's physical surface, only one of its approximations, that is: the geoid, in the case of the Stokes' approach or the telluroid -a surface in close proximity (of the order of ±100m) to the Earth's physical surface- in the Molodensky's approach respectively.

Theoretically, the Stokes' problem requires the knowledge of reduced (to the geoid) gravity anomalies which, in turn, requires the availability of levelling and gravity measurements (i.e. orthometric heights) all over the boundary surface. Respectively, in the Molodensky's approach the telluroid must be known a priori in order to reduce the measured surface gravity anomalies on it, i.e. to compute the corresponding gravity anomaly on the telluroid as

$$\Delta g = g(P) - \gamma(Q) \tag{5}$$

where g is the actual (measured) gravity at point P on the Earth's surface and Q is a point on the telluroid. Hence, in order to compute the normal gravity γ at the point Q on the telluroid one needs the corresponding normal height H^*. In practice, as the gravity anomaly values Δg must be known on the whole Earth for computing the height anomaly ζ, the length of the ellipsoidal normal between the Earth's surface and the telluroid, there are errors introduced in the computation of ζ because of the off-sets of the levelling datums.

Let's consider S to be the Earth's physical surface and W and $\boldsymbol{g}$ be, respectively, the actual geopotential and gravity vector on this surface. Then there exists a relation

$$\boldsymbol{g} = F(S, W), \quad (6)$$

that is, the gravity vector $\boldsymbol{g}$ on S is dependent on the geometry of surface S and the value of the geopotential W on it, and this dependence is expressed by F which is a nonlinear operator.

In the Molodensky's problem the task is to determine S, the Earth's surface, if $\boldsymbol{g}$ and W are given everywhere on it. Formally, we have to solve (6) for S

$$S = F_1(\boldsymbol{g}, W), \quad (7)$$

that is, to get geometry from gravity.

Nowadays, with the establishment of International Terrestrial Reference Frame (ITRF) and the development of the Global Satellite Navigation Systems (GNSS), we can determine very precisely the 3D positions of points on the Earth's surface that represents the boundary surface for the GBVP being considered. In this case, the geometry S is considered known, and we can now solve (6) for W

$$W = F_2(S, \boldsymbol{g}), \quad (8)$$

that is, to get potential from gravity.

In spite of the similarities between the two approaches, between getting geometry from gravity or getting potential from gravity, there exists a fundamental difference between them: equation (7) solves a free-boundary problem, since the boundary S covered with boundary data is taken a priori as unknown and 'free' to move only in the vertical direction, so that the information about the normal heights is already used a priori in order to fix the boundary, i.e. to obtain the telluroid. By contrast, equation (8) solves a fixed-boundary problem, since the boundary S is given, so that the realization of normal heights may be controlled by the independently determined quantities h and ζ. In mathematical terms, fixed-boundary problems are usually simpler than free ones.

Within the framework of BVP theory, the geoid determination problem is more suitably classified as an altimetry-gravimetry boundary value problem (AGBVP). The most important relevant formulations of AGBVPs or as they are discussed in the literature under the shorter name of 'Altimetry-Gravimetry Problems' (AGPs) are summarized in Table 1, where besides g and C, another observable at the points of measurements is considered, the geometric (ellipsoidal) heights h determined from precise GNSS positioning, and σ represents, in compact notation, the coordinate pair or solid angle (φ, λ).

Table 1. Basic formulations of AGBVPs

Part of the Earth's surface	Treatment of Parameters	Boundary Value Problem		
		AGP-I	AGP-II	AGP-III
Land	Known	g, σ, C	g, σ, C	g, σ, h
	Unknown	h	h	W
Sea	Known	σ, h, C	g, σ, h	σ, h, C
	Unknown	g	W	g

The type of AGP-I formulation, is a favourable approach for global or regional applications, whereby the ellipsoidal heights h being used are determined on the sea surface by satellite radar altimetry, when ship gravity data are not available or their coverage is poor. The AGP-II approach is often used in local areas close to coastlines where there is usually poor steric levelling data, but adequate coverage of ship gravity data and, when geopotential numbers on the sea surface are not available, ellipsoidal heights h are determined on the sea surface by satellite radar altimetry. The AGP-I and -II are free-boundary problems on land and fixed-boundary problems on sea. It has been pointed out in the geodetic literature, e.g. by Lehmann (2000), that the treatment of AGPs in spherical and constant radius approximation leads to mathematically well-posed problems in the case of the AGP-I and -II, while the AGP-I may exhibit features of ill-posedness in special situations. Well-posedness of AGPs is one of the most exciting (and still largely unsolved) problems in geodesy which is usually considered for mathematical analysis.

The AGP-III formulation is currently of interest for hybrid applications whereby, in the sea areas ellipsoidal heights h are determined by satellite altimetry, replacing sea gravity there, and on land, observed ellipsoidal heights h are determined by GNSS, replacing geometric levelling data. In contrast to the AGP-I and -II, the AGP-III is a fixed-boundary problem. Furthermore, this is generally a well-posed BVP, as shown in a more recent analysis on this formulation by Panou et al. (2011) and from previous numerical solutions presented by Čunderlík and Mikula (2009). Overall, the treatment of a fixed AGP formulation is considered as the most important for the near future, since, in practical terms, this would mean that height information on land could be provided entirely by space techniques rather than by the costly and time consuming conventional geometric levelling procedures.

In summary, considering the distinct features of the AGP-III, that is, being a fixed BVP, suitable of utilizing the data from the modern geodetic technologies (i.e. mixed), and being also a well-posed BVP, our approach to the height datum unification problem is based on the variant formulation outlined in the next section.

4. A variant formulation of a fixed mixed BVP

Realization of a unified global height datum, based on the joint processing of terrestrial and satellite geodetic data, admits a new variant formulation of the linear fixed mixed (altimetry-gravimetry) BVP. The linear fixed mixed problem can be mathematically described for each part of the Earth's surface by using the following form

$\nabla^2 T = 0$ in the 3D space outside the Earth's physical surface

$T = T^* + \delta W$ on sea

$\dfrac{\partial T}{\partial h} = -\,\delta g$ on land

$T = O(1/r)$ as $r \to \infty$

where ∇^2 is the Laplace operator, T is the (unknown) disturbing potential, $\delta g = g - \gamma$ denotes the gravity disturbances that correspond to difference between the measured gravity data on land (i.e. on the Earth's surface) and the normal gravity from a reference equipotential ellipsoid of rotation (e.g. GRS80) that can be computed at the same point by knowing its ellipsoidal height; T^* represents 'observed' values for the disturbing potential (e.g. from satellite altimetry, ship-borne gravimetry, etc. through the application of the well-known Bruns' formula) which requires the dynamic ocean topography to be removed e.g. by ocean levelling; δW is a perturbation of the Dirichlet boundary condition which, in this case, represents the datum disturbance parameter $\delta W = W_O - U_O$, that is, the difference between the actual (unknown) potential W_O and U_O, the normal potential on the surface of the reference ellipsoid (which is also used in the linearization process).

In practice, the value W_O of the actual gravity potential on the geoid represents a fundamental height datum parameter. Since W_O is not precisely known, the value U_O is not necessarily equal to the traditionally used theoretical or approximate values of W_O. The determination of a 'real world' W_O value (i.e. derived from global data) has not been considered until recently, when it was demonstrated by Sánchez (2008) that a reference geopotential value W_O can indeed be estimated from global satellite altimetry data and gravity disturbances obtained from a global Earth Gravity Model (EGM). This is a significant step forward, since the continuously improving modern geodetic techniques, especially those involving the precise determination of geometrical coordinates by GNSS positioning and satellite altimetry, and the accurate gravity field models provided by the new satellite missions, can now facilitate the accurate estimation of a suitable W_O value by evaluating powerful theoretical approaches that 30 years ago were not applicable in practice.

5. Outline of proposed method

The method outlined in this paper, as based on the previously described AGP formulation, can be explained with the simple example illustrated by Figure 1 which shows two equipotential surfaces defined by a reference station (fundamental stations I and II) in the two local height datums I and II respectively.

As long as we select the same reference ellipsoid, the quasigeoid determined by this method would make possible to establish a reference surface that contains middle and high frequency height components, but without reference to any local height datums. Therefore, the height anomalies ζ_0, as obtained from the solution of the previously described boundary value problem can be regarded as a 'global' height datum.

On the other hand, let us assume that in the local height datum I, for an arbitrary point A we know its normal height $H_{A,I}$. The local height anomalies $\zeta_{A,I}$ can be obtained by a combination of GPS/GNSS and levelling data

$$\zeta_{A,I} = h_A - H_{A,I} \tag{9}$$

where h_A denotes the ellipsoidal height obtained from GNSS procedures and $H_{A,I}$ corresponds to the normal height from levelling based on the local height datum I involved.

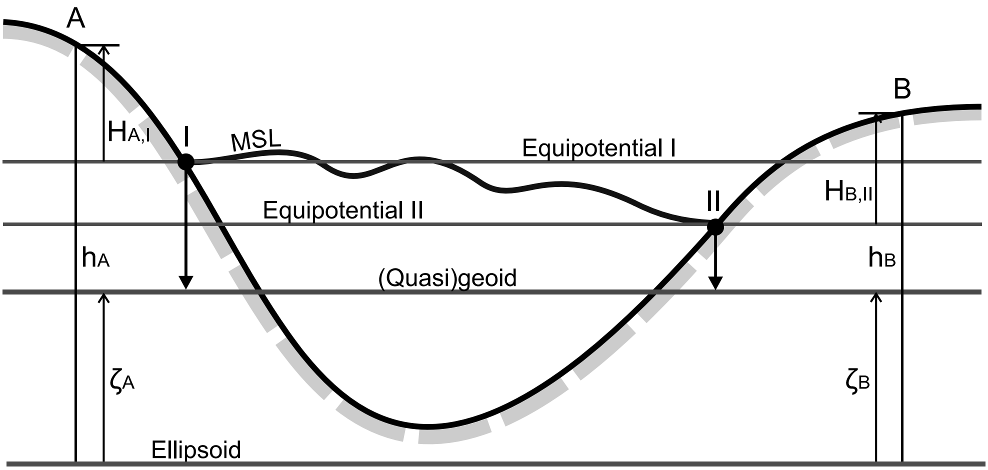

Figure 1. Height datum problem

If common ellipsoidal parameters are adopted for the computation of both local and global height anomalies, we obtain the following equation

$$\Delta W_I = \zeta_{A,I}\, \gamma_A - \zeta_{A0}\, \gamma_A = (\zeta_{A,I} - \zeta_{A0})\, \gamma_A = \Delta\zeta_{A,I}\, \gamma_A \tag{10}$$

where ΔW_I is the potential difference between the global and local height datum *I* and ζ_{A0} is the height anomaly for point *A* as obtained from the solution of the BVP. Note that, local height anomalies $\zeta_{A,I}$ and the height anomalies ζ_{A0} must correspond to same point *A* on the Earth surface. Similar equations to (9) and (10) hold for an arbitrary point *B* on local height datum *II*.

Considering the case of two local height datums, if we calculate their datum potential differences to the global datum individually, using eq. (10), the potential difference between two local height datums shall be given as

$$W_I - W_{II} = \Delta W_{I,II} = \Delta\zeta_{A,I}\,\gamma_A - \Delta\zeta_{B,II}\,\gamma_B \qquad (11)$$

where W_I and W_{II} represent the potential of the respective local height datums *I* and *II*, $\Delta W_{I,II}$ is the potential difference between the two local height datums and $\Delta\zeta_{A,I}$ and $\Delta\zeta_{B,II}$ represent the height differences between global height datum and local height datum at points *A* and *B* respectively.

In practice, this process would be applied to as many points on the local height datum *I*, in order to estimate a mean value $\Delta\overline{W}_I$ and its corresponding standard deviation for the potential difference between the global height datum and the local height datum *I*. Similarly, the process would be applied to as many points on the local height datum *II*, in order to estimate a mean value $\Delta\overline{W}_{II}$ and its corresponding standard deviation for the potential difference between the global height datum and the local height datum *II*. Finally, we can estimate the potential difference $\Delta\overline{W}_{I,II}$ between the two local height datums *I* and *II*. This same process can be applied for many local height datums, i.e. by applying eq. (10), and subsequently, the mutual relation between any pair of local height datums can be carried out by applying eq. (11). Therefore, a full unification can be realized in this truly integrated way.

6. Conclusions

In this paper we proposed the use of a fixed mixed BVP for attacking the classic height datum unification problem. The main advantage of this approach is that it is independent of any local height datum and that it makes use of all modern geodetic measurements (e.g. satellite altimetry at sea and GNSS-based geometric heights on land). The main outcome of the method is the potential differences between each local height datum with the global height datum realized through the solution of the aforementioned BVP that leads to the estimation of the quasigeoid. A comparison of potential differences from different height datums will then yield information on their relative vertical positions.

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