Time - dependent geodetic networks and the reference frame definition problem

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Abstract. The assessment and interpretation of the geodetic results regarding the detection of possible spatial displacements and the deformation parameters have to be combined with a realistic geophysical model for the area. Usually, this study is carried out by fitting the geodetic data to a polynomial function, which is considered sufficient to describe adequately the deformation pattern. In terms of the computational steps needed, this polynomial fitting can be accomplished (i) simultaneously by the analysis of the geodetic observations, in a dynamic adjustment, (ii) non simultaneously, in a sequential approach of the dynamic adjustment or (iii) by a simple comparison of the results between two epochs. The main intention of this article is to give a short description of all methods just mentioned, summarizing the existing methodologies that appear in the geodetic literature for crustal deformation studies. Emphasis is given on the analysis of time-dependent GNSS networks and on the reference frame definition problem.

Keywords. Deformation measurements, datum problem, GNSS networks, integrated approach, robust estimation, sequential adjustment.

1. Introduction

Various geodetic methods for the extraction of displacements and deformation parameters are recognized as useful techniques in many geophysical studies. Within the last thirty years, many geodetic applications have been presented in the literature. All these applications are based on the use of repeated observations over the geodetic network (properly established in the areas under investigation) and on the analysis of the results between different epochs of observations by means of appropriate models.

The assessment and interpretation of the geodetic results for the detection of possible spatial displacements and the deformation parameters have to be combined with a realistic geophysical model for the area. Usually, this study is carried out by fitting the geodetic data to a polynomial function, which is considered sufficient to describe adequately the deformation pattern. In terms of the computational steps needed, this polynomial fitting can be accomplished (i) simultaneously by the analysis of the geodetic observations, in a dynamic adjustment, (ii) non simultaneously, in a sequential approach of the dynamic adjustment or (iii) by a simple comparison of the results between two epochs.

Depending upon the kind and nature of the geodetic data being used (e.g. original observations or coordinates from a network adjustment), the corresponding mathematical model may suffer from some specific problems, such as the inconsistency of the reference frames, or the existence of non-positive covariance matrices.

The main intention of this work is to give a short description of all methods just mentioned, laying the emphasis on the reference frame definition problem, or on the non-positive covariance matrix problem, by referring to the robust estimation technique in every case.

The reference datum definition problem or datum problem or zero order design problem has received considerable attention since the pioneering work of Meissl (1965) and his famous "inner error theory" consisting in an instruction how to compute the suitable dispersion matrix of the point coordinates. Meissl's method has been popularized by Blaha (1971) and Pope (1973) who develop a powerful method of evaluating the pseudoinverse matrix uses the so called "solution space". The relation of various solutions to Meissl's inner constraints has been established with the introduction of the S-transformation by Baarda (1973). Subsequently all methods have been described in full length by Grafarend and Schaffrin (1974, 1976), Pelzer (1974), van Mierlo (1980), Koch (1982), Teunissen (1985). This problem dominated the geodetic literature in the 70s, although it still remains opportune in GNSS applications, as well as in the assessment of geodetic data for the detection of displacements and the estimation of deformation parameters.

In time dependent geodetic network a "common" reference frame for all epochs is needed. The definition of the reference frame in each epoch is based on analogous to the Meissl constraints, which are introduced for the coordinate differences in time. This procedure, that has been proposed by Pelzer (1971) for the geodetic applications of deformation measurements, is a discrete approximation to the definition of the reference frame under time-continues data (Dermanis, 2002). In the case of single adjustment per epoch the constraints are incorporated as inner constraints on the unknown corrections of approximate coordinates, by using common approximate coordinates for all epochs. A different approach of alternative constraints in the deformation networks can be found in Prescott (1981), Darby (1982) and Segall and Matthews (1988).

According to the geodetic literature there are three general methods, followed for the geodetic analysis of observations in time (Rossikopoulos, 2003, Dermanis and Kotsakis, 2006, Dermanis, 2009). We can call these methods: a) dynamic adjustment – the generalized approach, b) dynamic adjustment – the sequential approach (or adjustment in steps) and the third, the most famous one, c) comparison between two epochs.

2. Dynamic Adjustment. The pure geometric model

By the pure geometric model we can analyze classical observations (angles, distances, height differences) or modern observations (GNSS baselines, VLBI, SLR, etc). The observation equations for the *m* epochs are written in matrix form as

$$\mathbf{b} = \mathbf{A}\,\mathbf{x}_o + \mathbf{B}\,\mathbf{u} + \mathbf{D}\,\mathbf{y} + \mathbf{v} \tag{1}$$

where $\mathbf{x}_o$ is the vector of coordinate corrections for the reference epoch, and **u** is the vector of differences (the displacement vector), **y** are the nuisance parameters (e.g. the similarity transformation parameters for GNSS observations), **A**, **B** and **D** are the coefficient matrices and **v** the observational errors. The nuisance parameters **y** must be eliminated before applying the minimal constraints solution. In this way they do not participate in the optimality criteria for the reference datum definition problem (see Fritsch and Schaffrin 1981, for a more detail discussion about this issue).

The displacements may be treated as independent deterministic unknown parameters, in which case their relation to an underlying function is ignored. This approach has the advantage that it is free from any dubious assumptions about the structure of the underlying function, but the results of the simultaneous adjustment are equivalence to those of single adjustment per epoch. The dependence of displacements on underlying functions can be taken into account in two different ways. The first is to introduce a more or less empirical model for the function, which involves unknown parameters to be estimated from the adjustment of the observations. Typical choices are linear combinations of known base functions with unknown coefficients. Taking into account the analytical functions

$$\mathbf{u} = \mathbf{\Phi}\,\mathbf{a} \tag{2}$$

which are used to smooth-out the differential motions, representing the deformation model, the observation equations are written as

$$\mathbf{b} = \mathbf{A}\,\mathbf{x}_o + \mathbf{B}\mathbf{\Phi}\mathbf{a} + \mathbf{D}\,\mathbf{y} + \mathbf{v} \tag{3}$$

where **a** are unknown parameters an $\mathbf{\Phi}$ the matrix with elements depending on the known functions (the so called base functions) and the way than displacements depend on these functions, or

$$\mathbf{b} = \mathbf{A}\,\mathbf{x}_o + \mathbf{F}\,\mathbf{a} + \mathbf{D}\,\mathbf{y} + \mathbf{v} \tag{4}$$

The deformation model (2) can be a space model (e.g. tensors of deformations), a time model (e.g. rates of displacements) or a space-time model (e.g. rates of deformations).

In the first case, in which the displacements are considered as dependent on space only, we have for the point P_i

$$\mathbf{u}_i = \mathbf{\Phi}(\mathbf{x}_i)\,\mathbf{a} \tag{5}$$

where the unknown parameters **a** are common for all points at the same epoch. For two dimensions, a choice of analytic functions to describe the horizontal movements is

$$f_k(x,y) = a_{ko} + \sum_{l=1}^{r} \sum_{m=0}^{l} a_{kn}\, x^{l-m} y^m \ , \quad n = \frac{1}{2} l(l-1) + m \tag{6}$$

where n is the index counting of the term $x^{l-m} y^m$ and $k = 1, 2$ refers to the movements in x and y direction respectively. Relevant articles have been written by Margrave and Nyland (1980), Chrzanowski et al. (1983). In more detail, the displacements u_i and v_i of a point P_i, in x and y direction, are described by relations

$$u = a_{1o} + \sum_{l=1}^{r} \sum_{m=0}^{l} a_{1n}\ x^{l-m} y^m \ , \quad v = a_{2o} + \sum_{l=1}^{r} \sum_{m=0}^{l} a_{2n}\ x^{l-m} y^m \ . \tag{7}$$

Useful tools for creating analytic polynomial-type models of the form of the above relations result from the implementation methods of the theory of elasticity as the deformation of the Earth are associated with the classical case of continuous media mechanics. Such methods were used for the first time in the study of deformations of the Earth's crust in 1932 by Tsuboi (*Investigation on the deformation of the earth's crust in the Tango district connected with the Tango earthquake of 1927*, Bull. Eartquake Res. Inst.). He estimated the strain tensors from the temporal variation of point coordinates using the finite element method as proposed by Terada and Miyabe in their work *Deformation of the earth crust in Kwansai districts and its relation to the orographic feature* (Bull. Eartquake Res. Inst., 1929). For a more systematic study of deformation theory and its applications in geodesy and Geodynamics we refer to the work of Dermanis and Livieratos *Applications of deformation analysis in Geodesy and Geodynamics* (Rev. Geophys. Space Phys., 21(1), 41-50). Under the hypothesis of homogeneous deformation, the displacement of a point P_i is described by the equations

$$u_i = x_i' - x_i = x_i\, \varepsilon_{xx} + \frac{1}{2} y_i\, \gamma_2 - y_i\, \omega$$

$$v_i = y_i' - y_i = y_i\, \varepsilon_{yy} + \frac{1}{2} x_i\, \gamma_2 + x_i\, \omega \tag{8}$$

where $\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$, $\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$ (9)

are the principal strains across the north and east direction and

$$\gamma_2 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \ , \quad \omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \tag{10}$$

are the shear across any line parallel to east direction and the angle of infinitesimal rotation.

The deformation parameters in simultaneous adjustment of network observations of different epochs are usually treated as deterministic parameters, where the deformation of the entire network area or large part of it is considered homogeneous, or it is treated in conjunction with analytical interpolation methods. For example, combining relations (8), (9) and (10) with (7) the deformation parameters are resulting (Magrave and Nyland 1980, Bibby 1982)

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \sum_{l=1}^{r} \sum_{m=0}^{l} (l-m)\, a_{1n}\, x^{l-m-1} y^{m}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \sum_{l=1}^{r} \sum_{m=0}^{l} m\, a_{2n}\, x^{l-m} y^{m-1}$$

$$\gamma_2 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \sum_{l=1}^{r} \sum_{m=0}^{l} m\, a_{1n}\, x^{l-m} y^{m-1} + \sum_{l=1}^{r} \sum_{m=0}^{l} (l-m)\, a_{2n}\, x^{l-m-1} y^{m} \tag{11}$$

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(\sum_{l=1}^{r} \sum_{m=0}^{l} (l-m)\, a_{2n}\, x^{l-m-1} y^{m} - \sum_{l=1}^{r} \sum_{m=0}^{l} m\, a_{1n}\, x^{l-m} y^{m-1}\right)$$

The estimability problem of deformation parameters can be examined from the viewpoint of a particular solution of free network adjustment of two epochs of observations (Bibby, 1982, Chrzanowski et al., 1983, Xu et al. 2000) or from the viewpoint of the variations of strain parameters under similarity transformations (Dermanis, 1981, 1985b, Grafarend, 1985). In some way this problem was presented for the first time in 1966 by Frank in his work *Deduction of earth strains from survey data* (Bull. Seism. Soc. Amer. Vol. 56, 1), where the need for measuring distances to define the scale of the network and the size of the movement was emphasized.

The observations equations for the *m* epochs are

$$\begin{aligned}
\mathbf{b}_o &= \mathbf{A}_o\, \mathbf{x}_o + \mathbf{D}_o\, \mathbf{y}_o + \mathbf{v}_o \\
&\ \vdots \\
\mathbf{b}_\alpha &= \mathbf{A}_\alpha\, \mathbf{x}_o + \mathbf{D}_\alpha\, \mathbf{y}_\alpha + \mathbf{A}_\alpha\, \mathbf{\Phi}_\alpha\, \mathbf{a}_\alpha + \mathbf{v}_\alpha \\
&\ \vdots \\
\mathbf{b}_m &= \mathbf{A}_m\, \mathbf{x}_o + \mathbf{D}_m\, \mathbf{y}_m + \mathbf{A}_m\, \mathbf{\Phi}_m\, \mathbf{a}_m + \mathbf{v}_m
\end{aligned} \tag{12}$$

where $\mathbf{y}_\alpha$ are the nuisance parameters for each epoch t_α.

For the datum definition problem the minimal constraints are introduced in sequential form. The datum for the reference epoch is defined first, by applying the inner constrains

$$\dot{\mathbf{E}}\,\mathbf{x}_o = \mathbf{0} \tag{13}$$

and afterwards the datum for any epoch is defined, by convenient minimal constraints. Although we have the extended model, we must minimize the norm

$$\mathbf{u}_\alpha^T\,\mathbf{u}_\alpha = \min. \tag{14}$$

which refers to the displacements. This solution corresponds to the best fitting of the coordinates at the various epochs to a reference epoch. According to Fritsch and Schaffrin (1981) the nuisance parameters **y** must be eliminated before applying the minimal constraints solution. In this way they do not participate in the constraints. The minimal constraints, which correspond to optimality criterion (14), have the form

$$\mathbf{a}_\alpha^T\,\mathbf{\Phi}_\alpha^T\,\mathbf{\Phi}_\alpha\,\mathbf{a}_\alpha = \min. \quad \text{or} \quad \mathbf{a}_\alpha^T\,\mathbf{W}_\alpha\,\mathbf{a}_\alpha = \min. \tag{15}$$

The condition (14) is introduced in the minimal constraints form

$$\ddot{\mathbf{H}}_\alpha\,\mathbf{a}_\alpha = \mathbf{0} \tag{16}$$

where $$\ddot{\mathbf{H}}_\alpha = \ddot{\mathbf{E}}_\alpha\mathbf{W}_\alpha = \dot{\mathbf{E}}_\alpha\mathbf{\Phi}_\alpha = \sum_{i=1}^{N}\dot{\mathbf{E}}_i\,\mathbf{\Phi}_i \tag{17}$$

and $\ddot{\mathbf{E}}_\alpha$ is the inner constraints matrix

$$\ddot{\mathbf{E}}_\alpha\,\mathbf{a}_\alpha = \mathbf{0} \quad . \tag{18}$$

This matrix results from the relation

$$\ddot{\mathbf{E}}_\alpha = \dot{\mathbf{E}}_\alpha\mathbf{\Phi}_\alpha(\mathbf{\Phi}_\alpha^T\mathbf{\Phi}_\alpha)^{-1} \tag{19}$$

where $\dot{\mathbf{E}}_\alpha = [\,\dot{\mathbf{E}}_1\ \dot{\mathbf{E}}_2\ \ldots\ \dot{\mathbf{E}}_N\,]$ is the inner constraints matrix $\dot{\mathbf{E}}_\alpha\,\mathbf{x}_\alpha = \mathbf{0}$ for the N network points at epoch t_α.

Another choice for the reference datum definition problem is to minimize the "movement" in a particular direction, through minimal constraints. This “outer coordinate solution” was given by Prescott (1981) in a specific application where the relative movement, and therefore the direction to minimize the temporal differences of the coordinates, should be parallel to the direction of the fault. Darby (1982) generalized the outer coordinate solution by noting that the preferred direction can be different at different stations and in “model coordinate solution” of Segall and Matthews (1988) the displacement residuals are made as small as possible.

In the case that the displacements are considered to be dependent on time only, the deformation model has the simple form

$$\mathbf{u}_i = \mathbf{\Phi}(t,t')\,\mathbf{a}_i \quad . \tag{20}$$

If the movement is linear in time, we have the “velocity model”

$$u = x - x' = \delta t \frac{\partial u}{\partial t} = \delta t\, \dot{u} \ , \ \ v = y - y' = \delta t \frac{\partial v}{\partial t} = \delta t\, \dot{v} \qquad (21)$$

where $\delta t = t - t'$ is the time difference. The first application of this velocity model was presented by Morgan (1973) in his paper *Crustal Velocity and Strain*. Examples were given in Papo and Perelmuter (1983), Welsch (1986) and Vanicek et al. (1979), Mälzer et al. (1979) for vertical networks.

The observation equations for the m epochs are written in matrix form as

$$\begin{aligned}
\mathbf{b}_o &= \mathbf{A}_o\,\mathbf{x}_o + \mathbf{D}_o\,\mathbf{y}_o + \mathbf{v}_o \\
&\vdots \\
\mathbf{b}_\alpha &= \mathbf{A}_\alpha\,\mathbf{x}_o + \mathbf{D}_\alpha\,\mathbf{y}_\alpha + \mathbf{A}_\alpha\,\mathbf{\Phi}_\alpha\,\mathbf{a} + \mathbf{v}_\alpha \\
&\vdots \\
\mathbf{b}_m &= \mathbf{A}_m\,\mathbf{x}_o + \mathbf{D}_m\,\mathbf{y}_m + \mathbf{A}_m\,\mathbf{\Phi}_m\,\mathbf{a} + \mathbf{v}_m
\end{aligned} \qquad (22)$$

where the unknown parameters **a** are common for all epochs at the same point. The datum definition constraints are written as

$$\dot{\mathbf{E}}\,\mathbf{x}_o = \mathbf{0}\ , \ \ \ddot{\mathbf{H}}\,\mathbf{a} = \mathbf{0} \qquad (23)$$

where $\dot{\mathbf{E}}$ is the inner constraints matrix for the reference epoch and the matrix $\ddot{\mathbf{H}}$ is selected to minimize the norm

$$\mathbf{u}^T\,\mathbf{u} = \sum_{\alpha=1}^{m} \mathbf{u}_\alpha^T\,\mathbf{u}_\alpha = \mathbf{a}^T \left(\sum_{\alpha=1}^{m} \mathbf{\Phi}_\alpha^T\,\mathbf{\Phi}_\alpha \right) \mathbf{a} = \min. \qquad (24)$$

or $\mathbf{a}^T\,\mathbf{W}\mathbf{a} = \min.$, where $\mathbf{W} = \left(\sum_{\alpha=1}^{m} \mathbf{\Phi}_\alpha^T \mathbf{\Phi}_\alpha \right)$. (25)

Final, we have

$$\ddot{\mathbf{H}} = \ddot{\mathbf{E}}\,\mathbf{W} = \dot{\mathbf{E}} \sum_{\alpha=1}^{m} \mathbf{\Phi}_\alpha \qquad (26)$$

and the inner constraints $\ddot{\mathbf{E}}\,\mathbf{a} = \mathbf{0}$ matrix is

$$\ddot{\mathbf{E}} = \dot{\mathbf{E}} \left(\sum_{\alpha=1}^{m} \mathbf{\Phi}_\alpha \right) \left(\sum_{\alpha=1}^{m} \mathbf{\Phi}_\alpha^T \mathbf{\Phi}_\alpha \right)^{-1}. \qquad (27)$$

The minimal constraints take the form

$$\ddot{\mathbf{H}}\,\mathbf{a} = \sum_{i=1}^{N} \dot{\mathbf{E}}^i \left(\sum_{\alpha=1}^{m} \mathbf{\Phi}_\alpha^i \right) \mathbf{a}_i = \mathbf{0} \qquad (28)$$

For the velocity model $\mathbf{u} = \delta t_\alpha \, \dot{\mathbf{u}}$ the above equation becomes

$$\ddot{\mathbf{H}}\,\mathbf{a} = \left(\sum_{\alpha=1}^{m} \delta t_\alpha \right) \dot{\mathbf{E}}\,\dot{\mathbf{u}} = 0 \tag{29}$$

or $\dot{\mathbf{E}}\,\dot{\mathbf{u}} = \mathbf{0}$, as given in Papo and Perlemuter (1983).

When the displacements are considered to be dependent on space and time, the analytical deformation model takes the form

$$\mathbf{u}_i = \mathbf{\Phi}(\mathbf{x}_i, t, t')\,\mathbf{a} \tag{30}$$

where the unknown parameters a are common for all epochs and all points and the observations equations for the m epochs are the same as in (22). Examples are given in Bibby (1982), Snay et al. (1983, strain rate model) and Welsch (1986). For two dimensions, a choice of analytic functions to describe the horizontal movements, considered to be dependent on space and time, is

$$f_k(x, y, \delta t) = \sum_{l=1}^{m} a_{kl} \; \varphi_l(x, y, \delta t) \ . \tag{31}$$

A choice of functions f_k is the polynomial-type functions defined by the relation

$$f_k(x, y, \delta t) = \sum_{l=0} \sum_{m=0} \sum_{n=1} a_{kj} x^l y^m \delta t^n \tag{32}$$

where $j = j$ $(l,\ m,\ n)$ is the index counting of a_{kj} parameters. Another choice is the function of the type

$$f_k(x, y, \delta t) = f_{ok}(x, y) \;\; q(\delta t) \tag{33}$$

where the functions $f_{ok}(x, y)$ of position only have the form given above, and q is a function of time. The simultaneous adjustment of the observations of many epochs was presented in work *Geodetic networks versus time* (Whitten, 1967), where movements are approached by functions of space and time of this form. Subsequently relevant papers were given by Snay and Gergen (1978), Snay et al. (1983), Chrzanowski et all. (1986), Welsch (1986) and regarding the determination of the vertical movements by Holdahl (1978, 1980), Vanicek (1975), Vanicek et al. (1979), Holdahl and Hardy (1979) and Mälzer et al. (1979).

In equations (20) deformation process was seen as a change from an initial to a final state of the continuous media without considering the time elapsed between the two conditions. In many cases however it is important to know the rate at which these changes occur, especially when we have a continuous data stream and when the deformation is smooth in time. The displacement vector $\mathbf{u}_i$ of point P_i during the interval δt is given by the relation

$$\mathbf{u}_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \delta t \begin{bmatrix} x_i & 0 & \frac{1}{2} y_i & -y_i \\ 0 & y_i & \frac{1}{2} x_i & x_i \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_{xx} \\ \dot{\varepsilon}_{yy} \\ \dot{\gamma}_2 \\ \dot{\omega} \end{bmatrix} = \delta t \, \mathbf{\Phi}_i \, \dot{\mathbf{a}} \tag{34}$$

and the vector of "velocities"

$$\dot{\mathbf{u}}_i = \begin{bmatrix} \dot{u}_i \\ \dot{v}_i \end{bmatrix} = \begin{bmatrix} x_i & 0 & \frac{1}{2} x_i & -y_i \\ 0 & y_i & \frac{1}{2} y_i & x_i \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_{xx} \\ \dot{\varepsilon}_{yy} \\ \dot{\gamma}_2 \\ \dot{\omega} \end{bmatrix} = \mathbf{\Phi}_i \, \dot{\mathbf{a}} \tag{35}$$

Examples of application were given by Morgan (1973), Bibby (1973, 1975, 1982) and Welsch (1986).

The strain rate parameters can be calculated from the relations of the form (34) directly or combined with analytical interpolation methods. For example, combining equations (9), (10) and (33) for the description of movements, follows that

$$\dot{\varepsilon}_{xx} = \frac{\partial q}{\partial t} \sum_{l=1}^{r} \sum_{m=0}^{l} (l-m) \; a_{1n} \; x^{l-m-1} y^m$$

$$\dot{\varepsilon}_{yy} = \frac{\partial q}{\partial t} \sum_{l=1}^{r} \sum_{m=0}^{l} m \; a_{2n} \; x^{l-m} y^{m-1}$$

$$\dot{\gamma}_2 = \frac{\partial q}{\partial t} \sum_{l=1}^{r} \sum_{m=0}^{l} m \; a_{1n} \; x^{l-m} y^{m-1} + \sum_{l=1}^{r} \sum_{m=0}^{l} (l-m) \; a_{2n} \; x^{l-m-1} y^m$$

$$\dot{\omega} = \frac{1}{2} \frac{\partial q}{\partial t} \left(\sum_{l=1}^{r} \sum_{m=0}^{l} (l-m) \; a_{2n} \; x^{l-m-1} y^m - \sum_{l=1}^{r} \sum_{m=0}^{l} m \; a_{1n} \; x^{l-m} y^{m-1} \right) \tag{36}$$

The minimal constrains which satisfy the condition $\mathbf{u}^T \mathbf{u}$ for all epochs are written

$$\ddot{\mathbf{H}} \, \mathbf{a} = \left[\sum_{i=1}^{N} \dot{\mathbf{E}}^i \left(\sum_{\alpha=1}^{m} \mathbf{\Phi}_\alpha^i \right) \right] \mathbf{a} = \mathbf{0} \tag{37}$$

where $\dot{\mathbf{E}}^i$ is the submatrix of $\dot{\mathbf{E}}$ which corresponds to the point P_i and

$$\mathbf{\Phi}_\alpha^i = \mathbf{\Phi}(\mathbf{x}_i, t, t') \, . \tag{38}$$

Except for the method of Least Squares, robust parameter estimation with respect to outliers can be applied for all above models. Detailed discussion on these robust methods is given in Koch (1999).

3. Crustal Deformation Parameters on the Reference Ellipsoid from GNSS observations

Although the deformation of the crust occurs in three dimensions, there is a long tradition of two dimensional processing of data, mainly because of the classic combination of geodesy where the calculation of the horizontal position of points on the surface of the earth is separated from the calculation of the vertical, but also because of uncertainty problems due to of the shape and the lack of information resulting in the expansion of the deformation from the earth's surface, which made the observations, to the three-dimensional space.

Nowadays however, that the coordinates of points are determined in a three-dimensional reference system from GNSS observations, more and more authors try to analyze the deformation of the crust developing prediction techniques in three dimensions, such as finite elements with tetrahedral (Kiamehr and Sjoberg, 2005), or describing the deformation parameters in the three-dimensional geocentric reference system (Bruner, 1979, Reilly, 1982), or other alternative methods which take into account the three-dimensional nature of the deformations (Altimer, 1999, Voosoghi, 2000). Subsequently, the relations for the calculation of surface deformation parameters from observations of satellite systems will be given reduced on the surface of the reference ellipsoid and following the traditional separation of the horizontal deformation from the vertical movement.

Using the notation ε_{EE}, ε_{NN} instead of e_{xx}, e_{yy}, the time variations of longitude $\delta_t\lambda_i = \lambda_i(t') - \lambda_i(t) = \lambda_i' - \lambda_i$ and latitude $\delta_t\varphi_i = \varphi_i(t') - \varphi_i(t) = \varphi_i' - \varphi_i$ are given from the relation

$$\begin{bmatrix} \delta_t\varphi_i \\ \delta_t\lambda_i \end{bmatrix} = \begin{bmatrix} \varphi_i & 0 & \frac{1}{2}\frac{N_i\cos\varphi_i}{M_i}\lambda_i & -\frac{N_i\cos\varphi_i}{M_i}\lambda_i \\ 0 & \lambda_i & \frac{1}{2}\frac{M_i}{N_i\cos\varphi_i}\varphi_i & \frac{M_i}{N_i\cos\varphi_i}\varphi_i \end{bmatrix} \begin{bmatrix} \varepsilon_{NN} \\ \varepsilon_{EE} \\ \gamma_2 \\ \omega \end{bmatrix} \tag{39}$$

which expresses the projection of the three-dimensional deformation at point P_i on the reference ellipsoid. Similar relations were given by Pope (1966) and Snay and Cline (1980). In the above equation

$$N_i = \frac{a}{\sqrt{1 - e^2\sin^2\varphi_i}}, \quad M_i = \frac{a(1-e^2)}{\sqrt{(1 - e^2\sin^2\varphi_i)^3}} \tag{40}$$

are the radii of curvature of the first vertical and meridian section respectively, *a* is the large semi axis and *e* the prime eccentricity of the reference ellipsoid.

Various analytical functions (Holdahl, 1978, 1980, Holdahl and Hardy 1979) can be used to describe the vertical movement $\delta_t h_i$.

The variations $\delta_t\lambda_i$, $\delta_t\varphi_i$ can be described following the analytical prediction techniques and by polynomials of the form of the relations (6), which are rewritten as

$$\delta_t\varphi = a_{1o} + \sum_{l=1}^{r} \sum_{m=0}^{l} a_{1n}\, \varphi^{l-m} \lambda^m, \quad \delta_t\lambda = a_{2o} + \sum_{l=1}^{r} \sum_{m=0}^{l} a_{2n}\, \varphi^{l-m} \lambda^m \tag{41}$$

or, introducing and time parameters, according to the formulas (Snay and Gergen, 1978 and Snay at all.1983)

$$\delta_t\varphi = a_{1o} + \sum_{l=1}^{r} \sum_{m=0}^{l} a_{1n}\, \varphi^{l-m} \lambda^m\, \delta t^n$$

$$\delta_t\lambda = a_{2o} + \sum_{l=1}^{r} \sum_{m=0}^{l} a_{2n}\, \varphi^{l-m} \lambda^m\, \delta t^n \quad . \tag{42}$$

where n is the index counting of the term $x^{l-m} y^m$.

These relations are useful for determining the deformation tensors from GNSS observations in an extended adjustment model, or from the direct comparison of the temporal of GNSS observations in conjunction with the finite element method or other interpolation methods. For example, observation equations of GNSS baselines in curvilinear coordinates φ, λ, h are

$$\mathbf{r}_{ij}^{GPS} = \mathbf{r}_{ij}^{o} - \mathbf{R}_i^T \mathbf{G}_i^{1/2} \mathbf{z}_i + \mathbf{R}_j^T \mathbf{G}_j^{1/2} \mathbf{z}_j \tag{43}$$

where

$$\mathbf{R}_i = \mathbf{R}_1(90^o - \varphi_i)\mathbf{R}_2(90^o + \lambda_i) = \begin{bmatrix} -\sin\lambda_i & \cos\lambda_i & 0 \\ -\cos\lambda_i \sin\varphi_i & -\sin\lambda_i \sin\varphi_i & \cos\varphi_i \\ \cos\lambda_i \cos\varphi_i & \sin\lambda_i \cos\varphi_i & \sin\varphi_i \end{bmatrix} \tag{44}$$

$$\mathbf{G}^{1/2} = \begin{bmatrix} (N+h)\cos\varphi & 0 & 0 \\ 0 & M+h & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{45}$$

$\mathbf{G} = \mathbf{G}^{1/2}\mathbf{G}^{1/2}$ is the metric matrix such that $\mathbf{ds} = \mathbf{dr}^T\mathbf{dr} = \mathbf{dz}^T\mathbf{G}\,\mathbf{dz}$ and $\mathbf{z}_i$, $\mathbf{z}_j$ are the corrections of approximate coordinates φ^o, λ^o, h^o of two points. Then, the vector of the temporal variations $\delta_t\mathbf{r}_{ij} = \mathbf{r}'_{ij} - \mathbf{r}_{ij}$ of the components of the GPS base is given by

$$\delta_t\mathbf{r}_{ij} = -\mathbf{R}_i^T \mathbf{G}_i^{1/2} \mathbf{\Phi}_i\ \mathbf{a} + \mathbf{R}_j^T \mathbf{G}_j^{1/2} \mathbf{\Phi}_j\ \mathbf{a} \tag{46}$$

where we considered that $\delta_t\mathbf{z}_{ij} = \mathbf{\Phi}_{ij}\ \mathbf{a}$, or $\delta_t\mathbf{r}_{ij} = \left(\mathbf{B}_j\mathbf{\Phi}_j - \mathbf{B}_i\mathbf{\Phi}_i\right)\mathbf{a}$. (47)

The matrix $\mathbf{B}_i = \mathbf{R}_i^T\mathbf{G}_i^{1/2}$ has the analytical form

Table 1. A summary of modeling alternatives in four dimensional integrated geodesy (Dermanis and Rossikopoulos, 1988).

$\mathbf{w} = \mathbf{A}\,\mathbf{x}_o + \mathbf{B}\,\mathbf{u} + \mathbf{G}\,\mathbf{s}_o + \mathbf{D}\,\mathbf{q} + \mathbf{v}$ or $\mathbf{w} = \mathbf{A}\,\mathbf{x}_o + \mathbf{B}\,\mathbf{u} + \mathbf{G}\,\mathbf{s} + \mathbf{v}$	
treatment of displacements $\mathbf{u}$	treatment of variations of signals $\mathbf{q}$ (or $\mathbf{s}$) $\mathbf{s}_o$ always stochastic $\mathbf{s}_o \sim (\mathbf{0}, \mathbf{C}_{\mathbf{s}_o})$
analytical : $\mathbf{u} = \mathbf{\Phi}\,\mathbf{a}$	analytical : $\mathbf{q} = \mathbf{\Phi}\,\mathbf{b}$
stochastic : $\mathbf{u} \sim (\mathbf{0}, \mathbf{C}_\mathbf{u})$	stochastic : $\mathbf{s} \sim (\mathbf{0}, \mathbf{C}_\mathbf{s})$
space-stochastic & time-analytical: $\mathbf{u} = \mathbf{\Phi}\,\mathbf{a}$, $\mathbf{a} \sim (\mathbf{0}, \mathbf{C}_\mathbf{a})$	space-stochastic & time-analytical: $\mathbf{q} = \mathbf{\Phi}\,\mathbf{b}$, $\mathbf{b} \sim (\mathbf{0}, \mathbf{C}_\mathbf{b})$
analytical + stochastic: $\mathbf{u} = \mathbf{\Phi}\,\mathbf{a} + \delta\mathbf{u}$, $\delta\mathbf{u} \sim (\mathbf{0}, \mathbf{C}_{\delta\mathbf{u}})$	analytical + stochastic: $\mathbf{q} = \mathbf{\Phi}\,\mathbf{b} + \delta\mathbf{q}$, $\delta\mathbf{q} \sim (\mathbf{0}, \mathbf{C}_{\delta\mathbf{q}})$

$$\mathbf{B}_i = \begin{bmatrix} -(N_i + h_i)\sin\lambda_i\cos\varphi_i & -(M_i + h_i)\cos\lambda_i\sin\varphi_i & \cos\lambda_i\cos\varphi_i \\ (N_i + h_i)\cos\lambda_i\cos\varphi_i & -(M_i + h_i)\sin\lambda_i\sin\varphi_i & \sin\lambda_i\cos\varphi_i \\ 0 & (M_i + h_i)\cos\varphi_i & \sin\varphi_i \end{bmatrix} \tag{48}$$

or

$$\mathbf{B}_i = \begin{bmatrix} -N_i\sin\lambda_i\cos\varphi_i & -M_i\cos\lambda_i\sin\varphi_i \\ N_i\cos\lambda_i\cos\varphi_i & -M_i\sin\lambda_i\sin\varphi_i \\ 0 & M_i\cos\varphi_i \end{bmatrix} \tag{49}$$

if we will separate the deformation in a horizontal and a vertical part and assume that $h_i = 0$, case that is combined with the form given in the relation $\delta_t\mathbf{z}_i = \mathbf{\Phi}_i\,\mathbf{a}$. The above relations can be used to estimate the deformation parameters on the reference ellipsoid from GNSS observations.

4. The Integrated approach

The second way for taking into account the dependence of the displacements on the unknown function is the use of stochastic models in a computational process known as a model of integrated geodesy. The smooth structure of this function can be related to the similarity of function values, which are close in space and time, and the stochastic counterpart of similarity is correlation. The function is modeled as a stochastic process with known (usually zero) mean function and known covariance function in a model of Integrated Geodesy.

The model of integrated geodesy also includes gravity field parameters and their variations with the time as stochastic parameters, ignored or fixed to known values in pure geometrical adjustment. The connection of the stochastic characteristics of the displacements and those of the time variation of the gravity field can be taken under considerations.

All observations, including observations of the gravity field, for 3d Networks or for Vertical Networks can be analyzed simultaneously with an integrated model

$$\mathbf{b} = \mathbf{A}\,\mathbf{x}_o + \mathbf{B}\,\mathbf{u} + \mathbf{G}\,\mathbf{s} + \mathbf{v} \tag{50}$$

where $\mathbf{G}\,\mathbf{s}$ is the part of gravity field parameters. We can also make use of other geophysical information which is connected to the variation of the gravity field. Three-dimensional integrated geodesy in its various aspects has been treated by several authors, e.g. Hein (1986), Rossikopoulos (1986). Four-dimensional integrated geodesy has been treated in Collier et al. (1988), Hein (1984, 1986), Reilly (1981, 1982), Rossikopoulos (1986), Dermanis and Rossikopoulos (1988), Zhou Zhongmo and Chao Dingbo (1987).

The adjustment of combined GNSS, levelling and gravity networks is an interesting application of the integrated model. The incorporation of data of dynamic nature, such as gravity and height differences, simultaneously with GPS observations, involves the integrated approach, where the main advantage is a more reliable estimation of the vertical component of the deformation field. Application has been given by Hatjidakis and Rossikopoulos (2006).

Signals, i.e. the gravity field parameters and their variations in time, can be treated in a combined analytical-stochastic approach. All possibilities for the treatment of displacements and gravity signals, which can be further combined in all possible ways, are presented in table (1).

All models, resulting from the combination of different treatments of the signals, take the final form $\mathbf{b} = \mathbf{A}\,\mathbf{x} + \mathbf{G}\,\mathbf{s} + \mathbf{v}$ where **x** contains the deterministic parameters, **s** contains all the stochastic parameters and **v** are the observational errors. The adjustment problem is one of estimation with respect to **x** and prediction with respect to **s** and **v**. For the stochastic parameters it is assumed that their means

$$E\{\mathbf{s}\} = \boldsymbol{\mu}\,,\; E\{\mathbf{v}\} = \mathbf{0}$$

and the covariance matrices

$$E\{(\mathbf{s}-\boldsymbol{\mu})(\mathbf{s}-\boldsymbol{\mu})^T\} = \mathbf{C}_\mathbf{s}\,,\; E\{\mathbf{v}\mathbf{v}^T\} = \mathbf{C}_\mathbf{v}$$

are known, and $E\{(\mathbf{s}-\boldsymbol{\mu})\mathbf{v}^T\} = \mathbf{C}_{\mathbf{sv}} = \mathbf{0}$.

The estimates **x** and predictions **s** and **v** depend on the estimation and prediction principles being used. In the most usual case, the Best Linear Unbiased Estimation is used which is a special case of the “robust collocation solution” (Dermanis, 1991, 1993, Schaffrin, 1985).

5. Dynamic Adjustment. The Sequential Approach

In the rigorous sequential approach (stepwise adjustment), the adjusted coordinates of the single adjustments per epoch are the new observations. The final results must be equivalent to those of the generalized approach of dynamic adjustment model. It can be used according to the type of available data and the analysis follows three main steps:

a. Single adjustment per epoch.

This step includes the adjustment of the observations at each epoch, the statistical analysis and the final estimation of the coordinate set and its full covariance matrix at each epoch.

b. Best fitting of the coordinates at the various epochs to a reference epoch

The elimination of the difference between the coordinates at two distinct epochs t_α and t_β, which is due to their different datum definition, is obtained by the optimal fitting of the t_β coordinates to the corresponding t_α coordinates (at the reference epoch), applying the well known 2-d or 3-d similarity transformation.

c. Adjustment with a deformation model

The coordinates in the reference epoch and the transformed ones as described in the second step are adjusted taking into account a deformation model. The mathematical model for the all epochs is written in matrix form

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_o + \mathbf{\Phi}_1 \mathbf{a} + \mathbf{v}_1 \\ &\vdots \\ \mathbf{x}_\alpha &= \mathbf{x}_o + \mathbf{\Phi}_\alpha \mathbf{a} + \mathbf{v}_\alpha \\ &\vdots \\ \mathbf{x}_m &= \mathbf{x}_o + \mathbf{\Phi}_m \mathbf{a} + \mathbf{v}_m \end{aligned} \tag{51}$$

or $\mathbf{x} = \mathbf{x}_o + \mathbf{\Phi}\mathbf{a} + \mathbf{v}$ and $\mathbf{v}^T \mathbf{W} \mathbf{v} = \sum_{\alpha=1}^{m} \mathbf{v}_\alpha^T \mathbf{W}_\alpha \mathbf{v}_\alpha = \min.$ (52)

where $\mathbf{x}_o$ is the vector of coordinates corrections for the reference epoch, and the term $\mathbf{\Phi}\mathbf{a}$ describes the displacement vector $\mathbf{u}$. The displacements can be considered as dependent in time, or in space and time.

For the time span of the analysis period (all epochs), the corresponding covariance matrices $\mathbf{Q}$ derived from the transformations, are non-positive definite. In this case, any choice of generalized matrix $\mathbf{W} = \mathbf{Q}^-$ as a weight matrices, and therefore the pseudo inverse matrix $\mathbf{W} = \mathbf{Q}^+$, leads to the best unbiased estimations for the parameters $\mathbf{u}$ (Rossikopoulos, 2010).

The method is illustrated by Rossikopoulos et al. (1998), where GPS measurements are used associated to a 9-point geodetic network connecting the Greek and Italian coasts in the Ionian and Adriatic Sea.

6. The single solution. Comparison between two epochs

In the single solution the results from the separate adjustment of the network at two distinct epochs are analyzed. This is the most well received approach in the geodetic literature because of its simplicity. The algorithmic steps consist of:

a. A single adjustment per epoch.
b. The best fitting of the coordinates at the various epochs to a reference epoch.
c. The estimation of the deformations by one the following:
   - analytical interpolation methods (Brunner et all. 1981, Chrzanowski et all. 1983),
   - stochastical interpolation methods (Bencini et al. 1981, Dermanis et al. 1981),
   - finite element methods (Livieratos 1980, Dermanis and Grafarend 1992).

In smoothing interpolation where the coordinate variations are considered observations the initial equations and the criterion of least squares are

$$\mathbf{u} = \mathbf{\Phi}\,\mathbf{a} + \mathbf{e}\,,\;\; \mathbf{e}^T \mathbf{W}\,\mathbf{e} = \min. \tag{53}$$

where $\mathbf{W}$ is the weight matrix of $\mathbf{u}$. The covariance matrix $\mathbf{Q}$ of the displacements $\mathbf{u}$ derived from the transformations or from the inner constraint solutions, are non-positive definite. Any choice of generalized matrix $\mathbf{W} = \mathbf{Q}^-$ as a weight matrix, and therefore the pseudo inverse matrix $\mathbf{W} = \mathbf{Q}^+$ can be used. The pseudo inverse matrix can be derived as the “parallel sum” of the coefficient matrices $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$ of the normal equations of two epochs t_α και t_β (Zhong and Welsch, 1997)

$$\mathbf{W} = \mathbf{Q}^+ = \mathbf{N}_\alpha(\mathbf{N}_\alpha + \mathbf{N}_\beta)^- \mathbf{N}_\beta = \mathbf{N}_\beta(\mathbf{N}_\alpha + \mathbf{N}_\beta)^- \mathbf{N}_\alpha \tag{54}$$

According to Rao and Mitra (1971b) a unified theory of least squares, with the simple choice for the weight matrix $\mathbf{W} = (\mathbf{Q} + \mathbf{\Phi}\,\mathbf{U}\,\mathbf{\Phi}^T)^{-1}$, valid for all situations whether the variance-covariance matrix of observations $\mathbf{Q}$ is non-singular or not. This solution has also been proposed in geodetic literature by Bjerhammer (1973) and Uotila (1974). A simple choice of $\mathbf{U}$ in all situations is $\mathbf{U} = \delta^2\,\mathbf{I}$ and $\mathbf{W} = (\mathbf{Q} + \delta^2 \mathbf{\Phi}\,\mathbf{\Phi}^T)^{-1}$, where the coefficient δ^2 ($\delta \neq 0$) regulates the magnitude of the elements of the matrix $\mathbf{\Phi}\,\mathbf{\Phi}^T$ compared to the elements of matrix $\mathbf{Q}$.

In exact interpolation methods we have the problem of best fitting of polynomials to the coordinate variations and the observation model becomes

$$\mathbf{u} = \mathbf{\Phi}\,\mathbf{a} + \mathbf{e}\,,\;\; \mathbf{e}^T \mathbf{e} = \min. \tag{55}$$

Examples of robust estimation techniques in deformation models and in the analytical interpolation problems are given form Caspary and Borutta (1987), Zhong (1997) and Yang (1994), who gives robust estimation models for correlated observations.

The collocation method

$$\mathbf{u} = \mathbf{s} + \mathbf{v} \quad , \qquad \mathbf{v}^T \mathbf{W} \, \mathbf{v} + \mathbf{s}^T \mathbf{K}^{-1} \mathbf{s} = \min. \quad \text{or} \qquad \mathbf{s}^T \mathbf{K}^{-1} \mathbf{s} = \min. \tag{56}$$

where $\mathbf{u}$ is the "observation vector" after the remove of the datum difference effects and **W** is its weight matrix. For the creation of the covariance matrix **K** of "signals" **s** a selected covariance function must be used.

A first thought for the application of the collocation methods to geodetic data analysis for determining deformation is attributed to Elmer and Welsch (1981), although the application in photogrammetry for determining the deformation parameters of aerial images was preceded (Mikhail, 1976). At the same time, however, the method was applied to the analysis of geodetic data for the calculation of strain parameters in the area of Volvi in Greece (Dermanis et al., 1981) and Friuli in Italy (Bencicni et al., 1982). Applications to the estimation of the vertical movements were given by Hein and Kistermann (1981), Kanngieser (1983) and El-Fiky et all. (1997). Papers on horizontal and vertical movements (El-Fiky and Kato, 1999, Wu et all., 2006, Kahle et all., 1995), were followed.

A combined analytical-stochastic treatment is also possible for the displacements **u**. One part of the underlying function, the trend, is modeled with the help of a linear combination of known base functions and the remaining part is modeled as a stochastic process. As a result the signals **u** are replaced by $\mathbf{u} = \mathbf{\Phi} \, \mathbf{a} + \mathbf{s}$ where **a** are unknown deterministic parameters and **s** are stochastic variables.

At the final step of this we can include the input-output models, which are used to analyze the cause-effect relation of deformation processes and which are based on the condition that time series of the acting forces and the deformation are available. For describing this condition, special functions as multiple spectra, weighting functions and transfer functions are used. Applications and a rather complete list of literature may be found in Heine (1999).

A different approach for the uncertainty in deformation model fitting consists in description of the point coordinates for all epochs using fuzzy set formalism. The deformation parameters are considered as fuzzy numbers in a fuzzy regression model, which makes it possible to explicitly introduce the experts' opinion. The procedure produces fuzzy estimates, which separate the spatial uncertainty from the uncertainty in the model parameters and fuzzy statistical measures which reflect the probabilistic uncertainty of the interpolation. The reader is referred to Bardossy et al. (1988), Kacewitz (1994) and to Shyllon (2001) for application to geodetic data analysis. A method of fuzzy modeling of a deformation process was demonstrated by Heine (2001).

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