Anatomy of minimum constraints in geodetic network adjustment

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Abstract. The scope of this paper is to investigate the influence of the minimum constraints (MCs) on the reference frame parameters in a free-net solution. The non-estimable part of these parameters is analyzed in terms of its stability under a numerical perturbation of the constrained datum functionals. In practice, such a perturbation can be ascribed either to hidden errors in the known coordinates/velocities participating in the MCs, or to a simple change of their a priori values due to a datum switch on a different fiducial dataset. In addition, a perturbation of this type may cause a nonlinear variation to the estimable part of the reference frame parameters, since it theoretically affects the adjusted observations that are implied by the network's nonlinear observational model. The aforementioned effects have an impact on the quality of a terrestrial reference frame that is established via a minimum-constrained adjustment, and our study shows that they are both controlled through a characteristic matrix which is inherently linked to the MC system.

1. Introduction

The establishment of terrestrial reference frames (TRFs) is a fundamental task in geodesy, closely related to the zero-order design or datum choice problem of network optimization theory (Grafarend 1974, Teunissen 1985). Due to the inherent datum deficiency in all types of geodetic measurements, a set of external conditions is always required to obtain a unique and well defined TRF realization from a geodetic network adjustment. The use of minimum constraints signifies an optimal choice of such conditions in the sense that they provide the required information for the datum definition without interfering with the network's estimable characteristics. As a result, a minimum-constrained network is theoretically free of any geometrical distortion that could originate from the external datum conditions, while its estimable TRF parameters (if any) are determined solely from the available measurements without being affected by the user's minimum constraints. The latter affect only the non-estimable part of the reference frame parameters which is not reduced by the data, yet they influence the quality of the entire coordinate-based representation of the adjusted network (e.g. the covariance matrix of the estimated positions and their external reliability level).

A realized TRF through a network adjustment is subject to quality limitations

originating from the type of minimum constraints that are used for the datum definition. This is a well known fact to geodesists which is theoretically justified by the dependency of the covariance matrix of the estimated positions with regard to the selected minimum constraints. The propagated data noise on the realized TRF depends strongly on the chosen datum conditions, a fact that has been the foundation of the zero-order network optimization and motivated the formulation of the so-called inner constraints for geodetic network adjustment problems (Blaha 1971a, 1982). An equally important issue, which however has not been systematically investigated in the geodetic literature, is the *frame stability* that can be achieved from a network adjustment with a given set of minimum constraints. This represents a crucial aspect for the overall quality of a TRF realization and the objective comparison of different datum definition strategies. The basic question associated with this issue is not concerned with the propagated effect of the measurement errors into the adjustment results, but rather with the identification and quantification of the criteria under which a set of minimum constraints can provide a more stable TRF than another set of minimum constraints for the same network.

The main objective of this paper is to present a general framework for analyzing the frame stability in minimum-constrained networks. For this purpose, the influence of minimum constraints $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$ on the realized TRF will be studied via a 'perturbation analysis' for the network solution under a variation $\mathbf{dc}$ of the constrained datum functionals. In practice, such a variation can be attributed either to existing errors in the coordinates/velocities of the reference stations that participate in the datum conditions, or to a change of their a priori values due to a datum-switch into a different fiducial dataset (note that a well designed geodetic network should be fairly robust against such datum disturbances). Our analysis will show that a fundamental matrix always exists which characterizes the frame stability of any set of minimum constraints in a given network, and it can be used as a criterion matrix for an objective analysis of different datum definition strategies.

An important aspect that is also treated in our study is the geometrical distortion on a minimum-constrained network due to the aforementioned variation of the constrained datum functionals. This is an indirect nonlinear effect that remains hidden within the linearized framework of least-squares adjustment in rank-deficient nonlinear models, yet it theoretically exists and it can affect the estimable characteristics of a $\mathbf{dc}$-perturbed network solution. From a geodetic perspective, such an effect corresponds to a nonlinear propagation of datum related errors into the adjusted observations of a minimum-constrained network, and it may cause a degradation of their statistical accuracy that is implied by their formal covariance matrix.

2. Free networks and minimum constraints

2.1 General background

A $m \times m$ singular system of the so-called normal equations (NEQs)

$$\mathbf{N}\,(\mathbf{x} - \mathbf{x}_o) \; = \; \mathbf{u} \tag{1}$$

provides the fundamental setting for network adjustment problems and the establishment of spatial reference frames from terrestrial and/or space geodetic data. Typically, the above system is deduced from the linearized least squares (LS) inversion of a coordinate-based nonlinear parametric model ($\mathbf{y} = \mathbf{f}(\mathbf{x}) + \mathbf{v}$) that describes a noisy set of geodetic measurements in a local, regional or global network. Its analytic form depends on the rank-deficient Jacobian matrix $\mathbf{A} = \mathbf{f_x}(\mathbf{x}_o)$ of the network observables according to the well known relationships

$$\mathbf{N} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A} \quad \text{and} \quad \mathbf{u} = \mathbf{A}^{\mathrm{T}}\mathbf{P}\,(\mathbf{y} - \mathbf{f}(\mathbf{x}_o)) \tag{2}$$

where $\mathbf{x}_o$ is an initial approximation of the model parameters, $\mathbf{P}$ is the data weight matrix, and $\mathbf{x}$ is the unknown vector originating either from a static (coordinates only) or a dynamic (coordinates and velocities) modeling of the network stations with respect to an Earth-fixed reference system.

Any solution of Eq. (1) corresponds to what is commonly known as a *free-network (free-net) solution* (Sillard and Boucher 2001) and it holds a key role for the optimal analysis of a geodetic network on the basis of datum-deficient noisy measurements. Such solutions are theoretically equivalent to each other in the sense that they produce the same linearly adjusted observables $\hat{\mathbf{y}}$, and thus maintaining the same information about the network's estimable characteristics that are embedded in the given measurements. Their basic characteristic is that they provide unequivocal least-squares fit to the data vector $\mathbf{y}$ and they offer the standard framework for the realization of TRFs that directly reflect the data quality without being distorted by external datum-related biases.

The differences between free-net solutions are rigorously described through a linear transformation that depends on the TRF parameters which are not reduced (defined) by the geodetic observables in the underlying network. This transformation is known in the geodetic literature as *S-transformation* (Baarda 1973; van Mierlo 1980) and it provides a fundamental tool for the frame-based analysis in network adjustment problems.

2.2 Minimum constraints

The determination of a single solution of Eq. (1) requires a set of external conditions to specify a coordinate frame with respect to which the adjusted positions of the network stations shall be computed. A free-net solution is always associated with a set of k independent linear equations

$$\mathbf{H}\,(\mathbf{x} - \mathbf{x}_o) = \mathbf{c} \tag{3}$$

whose number is equal to the rank defect of the normal matrix ($k=m$-rank$\mathbf{N}$). These equations constitute the so-called *minimum constraints* (MCs) and they are theo-

retically satisfied by one, and only one, solution of the rank-deficient system $\mathbf{N}(\mathbf{x} - \mathbf{x}_o) = \mathbf{u}$. The terms $\mathbf{H}$ and $\mathbf{c}$ characterize completely a free-net solution and they provide the necessary information for its numerical computation either through a constrained LS estimator from the data vector $\mathbf{y}$, or through a S-transformation based on another solution of the same NEQ system.

From a theoretical perspective, the $k \times m$ matrix $\mathbf{H}$ needs to be of full-row rank and it has to satisfy the algebraic condition

$$\text{rank} \begin{bmatrix} \mathbf{N} \\ \mathbf{H} \end{bmatrix} = \text{rank} \mathbf{N} + \text{rank} \mathbf{H} = (m - k) + k = m \tag{4}$$

whereas the k-dimensional vector $\mathbf{c}$ is free to take any values within the column space of the matrix $\mathbf{H}$. The previous condition ensures the inversion of the extended NEQ system

$$(\mathbf{N} + \mathbf{H}^T \mathbf{W} \mathbf{H})(\mathbf{x} - \mathbf{x}_o) = \mathbf{u} + \mathbf{H}^T \mathbf{W} \mathbf{c} \tag{5}$$

which contains the (minimum) required information for the datum definition in terms of a pseudo-observation vector $\mathbf{c}$ that is associated with a design matrix $\mathbf{H}$ and a symmetric positive definite weight matrix $\mathbf{W}$. Given the condition in Eq. (4), the extended NEQs have a unique solution that satisfies both the original NEQ system (1) and the MC system (3), and it is independent of the weight matrix $\mathbf{W}$ (a fact that is sometimes overlooked in the geodetic literature)[1].

The theoretical freedom in the numerical selection of the auxiliary vector $\mathbf{c}$ could result in free-net solutions that are mathematically correct (in the sense that they satisfy both the singular NEQs and the imposed MCs) yet geodetically problematic due to the unreasonable magnitude of the estimated positions and/or the significant distortion in the geometrical characteristics of the linearly adjusted network; for some practical examples, see Xu (1997). In fact, even a MC vector $\mathbf{c}$ with arbitrarily small entries may have a 'distorting influence' on the free-net solution from Eq. (5) if a slight perturbation in its values is significantly amplified by the matrix $(\mathbf{N} + \mathbf{H}^T \mathbf{W} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{W}$. Note that this effect does not imply an ill-conditioned form for the constrained normal matrix $\mathbf{N} + \mathbf{H}^T \mathbf{W} \mathbf{H}$, yet it points to a frame-related instability for the adjusted network with respect to the adopted MCs.

2.3 Algebraic vs. geodetic admissibility of minimum constraints

The admissibility of the minimum constraints $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$, as stated in most

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[1] The independence of the solution of Eq. (5) from the weight matrix $\mathbf{W}$ does not hold if the system $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$ contains more equations than the network datum defect. In this case the solution of Eq. (5) will not generally satisfy Eq. (1) and Eq. (3); such 'over-constrained' adjustment schemes are not treated in this paper.

geodetic textbooks and related papers, relies on the fundamental condition (4) or on some of its algebraic equivalent forms that can be found in the relevant literature (e.g. Blaha 1971a, 1982; Koch 1999; Sillard and Boucher 2001). This algebraic admissibility depends on the structure of the matrices $\mathbf{H}$ and $\mathbf{N}$, and it is generally fulfilled by the datum implementation strategies that are used in geodetic practice (e.g. fixing a minimum number of station positions or applying inner constraints on the non-estimable TRF parameters over some or all of the network stations). A point of concern is the existence of degenerate network configurations that may cause a problematic adjustment for certain options of datum constraints due to the remaining rank deficiency in the extended NEQ system. Such special cases of singular MCs were investigated by Blaha (1971b) and later by Papo (1987), but it was Veis (1960) who first pointed out the possibility of singularities in the LS adjustment of satellite geodetic networks due to a geometrical faulty structure of the datum constraints.

An algebraically admissible set of minimum constraints guarantees the inversion of the augmented normal matrix $\mathbf{N} + \mathbf{H}^T \mathbf{W} \mathbf{H}$, yet it is not sufficient to secure a geodetically meaningful solution for the original datum-deficient NEQs. In order to better understand the meaning of this peculiar statement, it is helpful to clarify the role of the system $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$ in the context of free-net adjustment. The primary aim of this system is not the designation of any arbitrary coordinate frame, but the establishment of a coordinate frame in the neighbourhood of an existing frame that is realized by the approximate positions of the network stations. The need to refer a free-net solution into a TRF which is close to the one implied by the initial vector $\mathbf{x}_o$ stems from the linearization that is implicitly associated with the formation of the NEQ system in Eq. (1). An attempt to overcome this restriction was presented some years ago from Xu (1997) by assimilating into the singular NEQs the non-estimable TRF parameters of the network adjustment problem. Despite the theoretical interest of his over-parameterization approach, a free-net adjustment is (still) based on the logic of a 'linearized TRF implementation' whose practical significance is revealed from the example mentioned in the following remark.

Remark. The fixation of a minimum number of station coordinates to some a priori values does not (always) represent a valid datum definition scheme under a nonlinear treatment of the rank-deficient network. Nevertheless, such an option is usable for the linearized minimum-constrained adjustment as it leads to a unique datum specification relative to the frame of the initial station coordinates; a straightforward example is depicted in Fig. 1.

The constant vector of the MC system controls the closeness between the TRFs of the adjusted coordinates and the approximate coordinates at the network stations, on the basis of a minimum number of 'datum functionals' $\mathbf{c}$. It is critical, though, that this term does not cause any detectable disturbance on the true (nonlinear) geometrical characteristics $\hat{\mathbf{y}} = \mathbf{f}(\hat{\mathbf{x}})$ of the adjusted network and the TRF parameters that are already reduced by the available data. Moreover, a small variation of
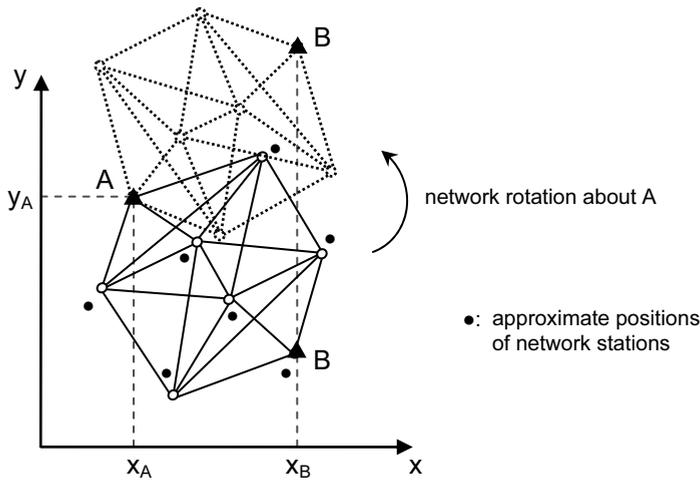
Fig. 1. The minimum constraints x_A=*const.*, y_A=*const.* and x_B=*const.* for the adjustment of a horizontal trilateration network do not theoretically yield a unique datum definition, since they cannot distinguish between the two symmetrical solutions that are shown in the above figure. However, a unique adjusted solution is practically obtained through these constraints, which is the one that lies closer to the approximate coordinates of the network stations.

the elements of **c** (e.g. due to coordinate/velocity errors at the reference stations that participate in the datum constraints) should not spawn an instability in the TRF of the adjusted network, neither interfere with its estimable characteristics. These are the main aspects behind a geodetically meaningful free-net solution which cannot be guaranteed by an admissible MC matrix **H**, as they are directly influenced by the MC vector **c** and the sensitivity of the constrained NEQs with respect to its disturbance. From a geodetic perspective the MC vector cannot take any values within the range of the MC matrix, a fact that creates a convoluted dependence among the basic components of the free-net adjustment problem.

Let us give a didactic example concerning the minimum-constrained adjustment of a horizontal trilateration network based on the fixation of three coordinates over its stations, namely x_A, y_A and x_B (see Fig. 2). In this case, the terms **H** and **c** have the general form:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} \tilde{x}_A - x_A^o \\ \tilde{y}_A - y_A^o \\ \tilde{x}_B - x_B^o \end{bmatrix} \tag{6}$$

where x_A^o, y_A^o, x_B^o are the approximate coordinate values, and $\tilde{x}_A$, $\tilde{y}_A$, $\tilde{x}_B$ denote the fixed coordinate values which will jointly define the TRF origin and orientation
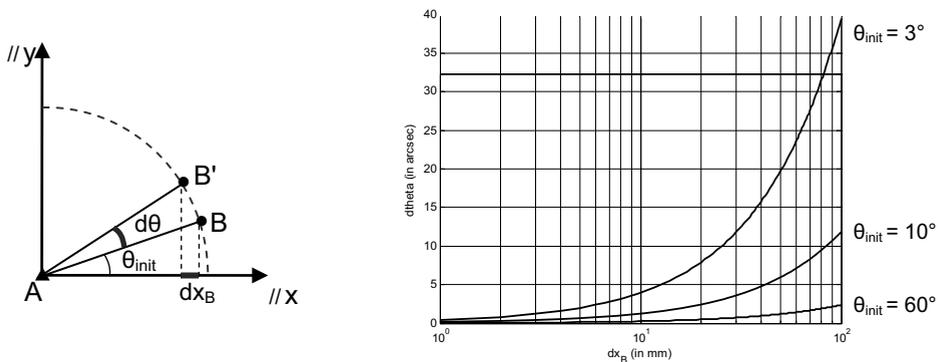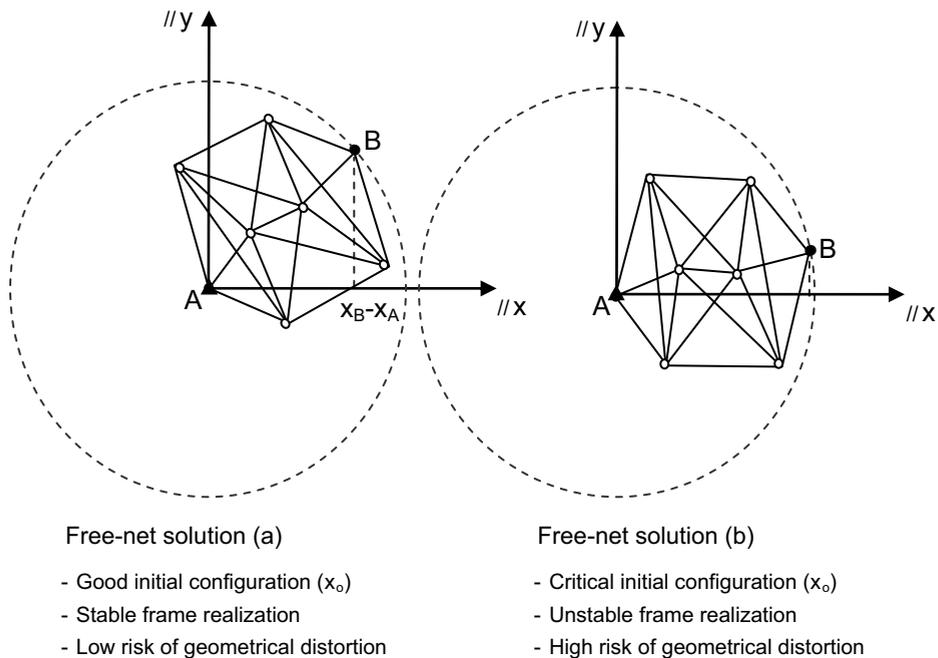
Fig. 2. A schematic description for the distorting effect that the minimum constraints may cause on the free-net solution of a horizontal trilateration network. The datum conditions refer to the fixation of the *x* and *y* coordinates of point A and the *x* coordinate of point B. If the difference of the fixed *x*-coordinates exceeds a critical value then a distortion will occur in the geometrical form $\hat{\mathbf{y}} = \mathbf{f}(\hat{\mathbf{x}})$ of the linearly adjusted network (in such case a convergent LS solution cannot be achieved through an iterative adjustment scheme). The lower two graphs depict the orientation disturbance of the minimum-constrained solution under a small change in the fixed *x*-coordinate of B (the numerical graph is based on an adjusted geometrical distance between A and B of 10 km).

of the horizontal network. In the absence of any geometrical configuration defect, the matrix **H** will fulfill the condition (4) and it will impose a valid datum definition for the linearly adjusted network. However, if the value $\tilde{x}_B$ (or, more pre-

cisely, the value of the difference $\tilde{x}_B - \tilde{x}_A$) exceeds a certain limit then the free-net solution will be deformed, thus affecting the network scale that is implicitly defined through the distance measurements (see Fig. 2). The initial configuration of the network stations ($\mathbf{x}_o$) plays a key role in quantifying the threshold for the reference coordinates' variation that could cause such a problematic solution. In the particular example it is evident that, as the point B lies closer to the x axis, the adjusted network could be effectively distorted under smaller disturbances of the MC vector. Note that even if the difference $\tilde{x}_B - \tilde{x}_A$ does not exceed a critical limit, the orientation of the free-net solution becomes increasingly unstable in this case (see Fig. 2).

Concluding this section, we need to make a final comment in view of the nonlinear character of LS network adjustment problems. Since a free-net solution is practically determined through an iterative adjustment scheme, any set of k datum conditions $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$ yielding a convergent LS estimate should always lead to the same geometrical form $\hat{\mathbf{y}} = \mathbf{f}(\hat{\mathbf{x}})$ for the adjusted network (note, however, that a rigorous convergence analysis for the linearized LS solution in rank-deficient nonlinear models does not currently exist in the geodetic literature). The crucial point to be emphasized here is that a convergent free-net solution does not necessarily realize a stable TRF over the network stations, and its geometrical characteristics may be affected under a small perturbation $\mathbf{dc}$ of the constrained datum functionals. These important issues are schematically described in Fig. 2 for the simple case of a horizontal network, and they will be analyzed under a more general setting in Sect. 4.

3. Mathematical background

A number of important algebraic formulae that are relevant to the free-net adjustment problem are reviewed in this section. Our presentation gives only an overview of the required material for the TRF stability analysis of the next sections, without focusing on mathematical details but rather outlining the essential theoretical tools for the purpose of this paper.

3.1 Basic relationships

The general solution of a singular NEQ system $\mathbf{N}(\mathbf{x} - \mathbf{x}_o) = \mathbf{u}$ can be expressed by the formula

$$\hat{\mathbf{x}} = \mathbf{x}_o + \mathbf{N}^-\mathbf{u} + (\mathbf{I} - \mathbf{N}^-\mathbf{N})\mathbf{z} \tag{7}$$

where $\mathbf{N}^-$ is a generalized inverse of the normal matrix $\mathbf{N}$, and $\mathbf{z}$ is an arbitrary vector. The above expression is valid in view of the fundamental property $\mathbf{N}\mathbf{N}^-\mathbf{A}^T = \mathbf{A}^T$ that applies when $\mathbf{N} = \mathbf{A}^T\mathbf{P}\mathbf{A}$ (Koch 1999, p. 51).

The primary link of Eq. (7) with the formulation of the free-net adjustment problem is rooted in the basic formula:

$$\mathbf{N}^{-} \;=\; (\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1} \tag{8}$$

which gives the generalized inverse of a symmetric semi-positive definite matrix $\mathbf{N}$ in terms of a full-row rank matrix $\mathbf{H}$ that satisfies the algebraic condition (4) and an arbitrary symmetric positive definite matrix $\mathbf{W}$.

When the NEQs generalized inverse originates from Eq. (8) then the condition $\mathbf{H}\mathbf{N}^{-}\mathbf{A}^{\mathrm{T}} = \mathbf{0}$ is always fulfilled. Consequently, by multiplying both sides of Eq. (7) with the matrix $\mathbf{H}$, we deduce that the general NEQs solution complies with a system of independent linear equations

$$\mathbf{H}(\hat{\mathbf{x}} - \mathbf{x}_{\mathrm{o}}) \;=\; \mathbf{H}\mathbf{z} \tag{9}$$

which corresponds to the required MCs for the datum definition in a free-net solution. Note that the MC vector is now identified as $\mathbf{c} = \mathbf{H}\mathbf{z}$, a fact that reveals an important issue which was implied in our previous discussions: the minimum constraints should form a consistent linear system and thus their constant vector must belong to the range of the MC matrix. Fortunately, the way that the MC vector is numerically constructed in the geodetic practice conforms to such a mathematical requirement, as it will be explained later in the paper.

The weight matrix $\mathbf{W}$ that is used in the computation of $\mathbf{N}^{-}$ does not affect the NEQs solution in Eq. (7) and it does not interfere with the validity of the minimum constraints in Eq. (9). A free-net solution remains therefore independent of the weighting with which the MCs are implemented into the LS adjustment algorithm; however, its statistical accuracy assessment may account for the prior uncertainty of the MC vector (see also Sect. 5).

For any $m \times m$ NEQ system with rank defect $k = m$-rank$\mathbf{N}$ there exists a class of $k \times m$ full-row rank matrices $\mathbf{E}$ with the fundamental property (Blaha 1971a, Koch 1999)

$$\mathbf{A}\mathbf{E}^{\mathrm{T}} = \mathbf{0} \quad \text{and thus} \quad \mathbf{N}\mathbf{E}^{\mathrm{T}} = \mathbf{0} \tag{10}$$

These matrices are identified in this paper as *type-E matrices* and they hold a crucial role in network adjustment problems. Any type-E matrix is an algebraically admissible MC matrix that generates the so-called *inner constraints* $\mathbf{E}(\mathbf{x} - \mathbf{x}_{\mathrm{o}}) = \mathbf{c}$ with well known optimal statistical properties for the corresponding solution of the NEQ system. For more details, see also Blaha (1982), van Mierlo (1980), Papo and Perelmuter (1981).

In the context of our present study, the following equations are of particular importance:

$$(\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{N} \;=\; \mathbf{I} \;-\; \mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{H} \tag{11}$$

$$(\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H} \;=\; \mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{H} \tag{12}$$

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where $\mathbf{H}$ corresponds to any MC matrix that can be associated with the original NEQ system. The above expressions define the fundamental projector matrices which are used in the formulation of the S-transformation, as described in the next section.

3.2 S-transformation

The *S-transformation* is a key tool that relates different free-net solutions of the same singular NEQ system. In its simplest form, it can be expressed by the formula:

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}' + \mathbf{E}^T \boldsymbol{\theta} \tag{13}$$

where $\mathbf{E}$ is a full-row rank matrix that satisfies the fundamental property (10). The vector $\boldsymbol{\theta}$ reflects the degrees of freedom in the inversion of the rank-deficient normal matrix $\mathbf{N}$, and it quantifies the difference between free-net solutions through k 'datum transformation parameters'.

Since there are infinitely many type-E matrices, the transformation parameters $\boldsymbol{\theta}$ are not uniquely defined and they depend on the choice of $\mathbf{E}$ that appears in Eq. (13). Their values can be determined in a straightforward way by multiplying both sides of the previous equation with an *arbitrary MC matrix* and then solving for $\boldsymbol{\theta}$, in which case we get:

$$\boldsymbol{\theta} = (\tilde{\mathbf{H}}\mathbf{E}^T)^{-1}\tilde{\mathbf{H}}(\hat{\mathbf{x}} - \hat{\mathbf{x}}') \tag{14}$$

The above result is invariant with respect to the MC matrix $\tilde{\mathbf{H}}$ provided that both vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ correspond to distinct solutions of the same NEQ system.

Based on Eq. (14), the forward S-transformation may also be expressed by the equivalent model

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}' + \mathbf{E}^T(\tilde{\mathbf{H}}\mathbf{E}^T)^{-1}\tilde{\mathbf{H}}(\hat{\mathbf{x}} - \hat{\mathbf{x}}') \tag{15}$$

where $\tilde{\mathbf{H}}$ denotes again an arbitrary MC matrix. If the S-transformed vector $\hat{\mathbf{x}}$ needs to satisfy a particular set of minimum constraints $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$, then Eq. (15) takes the form:

$$\hat{\mathbf{x}} = (\mathbf{I} - \mathbf{E}^T(\mathbf{H}\mathbf{E}^T)^{-1}\mathbf{H})\hat{\mathbf{x}}' + \mathbf{E}^T(\mathbf{H}\mathbf{E}^T)^{-1}(\mathbf{c} + \mathbf{H}\mathbf{x}_o) \tag{16}$$

The last equation provides the fundamental basis for analyzing the effect of the MC vector $\mathbf{c}$ (and its disturbance) on the geodetic admissibility of a free-net solution that is compliant with a given MC matrix $\mathbf{H}$.

3.3 'Helmertization' of S-transformation

For every singular NEQ system there exists a type-E matrix which is independent of the data weighting and it depends only on the datum defect and the spatial con-

figuration of the network stations ($\mathbf{x}_o$). The corresponding S-transformation parameters $\boldsymbol{\theta}$ admit a straightforward interpretation and they reflect (the differences of) the non-estimable TRF parameters in $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ due to the different datum conditions that were used in each solution.

The aforementioned matrix is formally known as the *inner-constraint matrix* and it stems from the linearized Helmert transformation that describes a differential similarity between 'nearby' Cartesian coordinate frames over an N-point network (Blaha 1971a, p.23)

$$\mathbf{x}_{TRF} = \mathbf{x}'_{TRF'} + \mathbf{G}^{T}\mathbf{q} \tag{17}$$

where the Helmert transformation matrix $\mathbf{G}$ is

$$\mathbf{G} = \begin{array}{c} t_x \\ t_y \\ t_z \\ \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \delta s \end{array} \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 & 0 & 1 \\ 0 & z'_1 & -y'_1 & \cdots & \cdots & 0 & z'_N & -y'_N \\ -z'_1 & 0 & x'_1 & \cdots & \cdots & -z'_N & 0 & x'_N \\ y'_1 & -x'_1 & 0 & \cdots & \cdots & y'_N & -x'_N & 0 \\ x'_1 & y'_1 & z'_1 & \cdots & \cdots & x'_N & y'_N & z'_N \end{bmatrix} \tag{18}$$

and the vector $\mathbf{q}$ contains the seven parameters of the linearized similarity transformation, namely three translations (t_x, t_y, t_z), three small rotation angles $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$ and one differential scale factor (δs).

The inner-constraint matrix $\mathbf{E}$ consists of the particular rows of $\mathbf{G}$ that correspond to the non-estimable TRF parameters of the observed network. In case of dynamic networks, where both coordinates and velocities need to be jointly estimated from a combined adjustment of time-dependent data, the matrix $\mathbf{G}$ (and also $\mathbf{E}$) should be expanded to include additional rows for the rates of the TRF parameters according to the time-varying similarity transformation model; see Sillard and Boucher (2001), Altamimi et al. (2002), Soler and Marshall (2003).

The Helmert matrix $\mathbf{G}$ and the parameter vector $\mathbf{q}$ of the linearized similarity transformation model, for a given network, can be generally decomposed as:

$$\mathbf{G} = \begin{bmatrix} \mathbf{E} \\ \hline \overline{\mathbf{E}} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} \boldsymbol{\theta} \\ \hline \overline{\boldsymbol{\theta}} \end{bmatrix} \tag{19}$$

where $\mathbf{E}$ is the inner-constraint matrix that refers to the non-estimable TRF parameters $\boldsymbol{\theta}$ of the underlying network, and $\overline{\mathbf{E}}$ is the complement matrix corresponding to the estimable TRF parameters $\overline{\boldsymbol{\theta}}$ that are inherently defined through the network observations.

4. MC-perturbation analysis of free networks

The behaviour of a free-net solution under a perturbation of its associated MCs reflects (an important part of) the TRF quality that can be achieved through a geodetic network adjustment. Our aim in this section is to study the above effect and to expose any problems related to the geodetic admissibility of a set of MCs for a given network.

4.1 Effect on the network's non-estimable characteristics

Let $\hat{\mathbf{x}}$ be a free-net solution of a singular NEQ system that is compliant with a particular set of minimum constraints, namely $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$. The disturbance of such a solution due to a variation of the constant vector $\mathbf{c}$ is given from the expression:

$$\mathbf{d}\hat{\mathbf{x}} \;=\; \mathbf{E}^T(\mathbf{H}\mathbf{E}^T)^{-1}\mathbf{dc} \tag{20}$$

which is obtained by differentiating the S-transformation formula in Eq. (16). Alternatively, the previous equation may be derived through the differentiation of the free-net solution from the extended NEQs in Eq. (5) taking also into account the fundamental relationship in Eq. (12).

The induced disturbance of the non-estimable TRF parameters in the adjusted network is

$$\mathbf{d}\boldsymbol{\theta} \;=\; (\mathbf{H}\mathbf{E}^T)^{-1}\mathbf{dc} \tag{21}$$

and it essentially corresponds to the S-transformation parameters between the initial solution (based on $\mathbf{H}$ and $\mathbf{c}$) and the disturbed solution (based on $\mathbf{H}$ and $\mathbf{c}+\mathbf{dc}$). Note that the auxiliary matrix $\mathbf{W}$ does not influence any of the previous terms due to the algebraic insensitivity of free-net adjustment with respect to the MCs weight matrix.

The last equation is of fundamental value for the analysis of free networks as it dictates the influence of the MCs to each of the non-estimable frame parameters. The inverse of the square matrix $\mathbf{H}\mathbf{E}^T$ controls the datum sensitivity in the network solution and it has a key role for the frame stability in the presence of a perturbation (error) in the selected minimum constraints. This matrix is not necessarily diagonal, a fact that signifies that each constraint may affect more than one, or even all, of the non-estimable TRF components in the adjusted network.

The geodetic admissibility of the minimum constraints $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$ is influenced by the form of the matrix $(\mathbf{H}\mathbf{E}^T)^{-1}$. Depending on the numerical structure of this matrix, an 'unstable' free-net solution could emerge through Eq. (5) or (16) in the sense that a small error in the datum conditions may corrupt significantly not only the non-estimable TRF parameters but also the estimable information that is contained in the linearly adjusted observations (see next section). The occurrence of

this unfavorable effect depends on the spatial configuration of the underlying network in tandem with the type of its datum deficiency and the 'geometry' of the selected MCs, all of which are reflected into the *TRF stability matrix* $(\mathbf{HE}^T)^{-1}$.

As an example, let us recall the simple case of the horizontal trilateration network given in Fig. 2. The minimum constraints in this example refer to the fixation of the three coordinates x_A, y_A, x_B, and they lead to the following matrix expressions:

$$\mathbf{HE}^T = \begin{bmatrix} 1 & 0 & y_A^o \\ 0 & 1 & -x_A^o \\ 1 & 0 & y_B^o \end{bmatrix} \tag{22}$$

and

$$(\mathbf{HE}^T)^{-1} = \frac{1}{y_A^o - y_B^o} \begin{bmatrix} -y_B^o & 0 & y_A^o \\ x_A^o & y_A^o - y_B^o & -x_A^o \\ 1 & 0 & -1 \end{bmatrix} \tag{23}$$

If the azimuth between the datum points A and B is close to $\pm 90°$ ($y_A^o \approx y_B^o$) then a LS adjustment with respect to an 'unstable' datum will take place from which both the TRF origin and orientation will be weakly defined. Note that similar problems may also arise in other types of static or dynamic 2D/3D networks whose datum definition is associated with a 'problematic' matrix $(\mathbf{HE}^T)^{-1}$.

4.2 Effect on the network's estimable characteristics

The estimable characteristics of a free network are rendered into two basic components: the vector of the adjusted observations and the TRF parameters that are inherently defined from the available measurements. Theoretically, both of these components remain invariant after a numerical perturbation (**dc**) of the MCs or, more generally, under any S-transformation applied to the estimated vector $\hat{\mathbf{x}}$. This property is valid within the linearized framework of LS inversion in rank-deficient nonlinear models, yet it does not provide an exact assessment of the distortionless behavior in any MC solution. Some general aspects about the potential distortion of the estimable characteristics of free networks will be now outlined.

Adjusted observations

The vector of the linearly adjusted observations from a LS network adjustment is given by the formula

$$\begin{aligned} \hat{\mathbf{y}} &= \mathbf{f}(\mathbf{x}_o) + \mathbf{A}(\hat{\mathbf{x}} - \mathbf{x}_o) \\ &= \mathbf{f}(\mathbf{x}_o) + \mathbf{f_x}(\mathbf{x}_o)(\hat{\mathbf{x}} - \mathbf{x}_o) \end{aligned} \tag{24}$$

Considering an iterative implementation of the network's adjustment algorithm (where the approximate vector $\mathbf{x}_o$ is replaced at each step by the previously estimated position vector $\hat{\mathbf{x}}$ until a satisfactory convergence is achieved), the above estimate after sufficient iterations is practically compatible with the original nonlinear observational model, that is $\hat{\mathbf{y}} \simeq \mathbf{f}(\hat{\mathbf{x}})$.

The coordinate-based disturbance of the adjusted observations is expressed as:

$$\mathbf{d}\hat{\mathbf{y}} = \mathbf{A}\,\mathbf{d}\hat{\mathbf{x}} \tag{25}$$

which, in view of Eq. (20), becomes

$$\mathbf{d}\hat{\mathbf{y}} = \mathbf{A}\mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{dc} = \mathbf{0} \tag{26}$$

thus confirming the invariance of the adjusted observations under a MC disturbance within the free-net adjustment.

The previous property holds only to a first-order approximation of the observational model since it neglects the contribution of its nonlinear terms to the variation of the network observables. Based on a second-order approximation, for example, we would have that

$$
\begin{aligned}
\hat{\mathbf{y}}_{(\mathbf{H},\,\mathbf{c}+\mathbf{dc})} &= \mathbf{f}(\hat{\mathbf{x}}+\mathbf{d}\hat{\mathbf{x}}) \\
&\simeq \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{A}\,\mathbf{d}\hat{\mathbf{x}} + \boldsymbol{\xi} \\
&= \hat{\mathbf{y}}_{(\mathbf{H},\,\mathbf{c})} + \underbrace{\mathbf{A}\,\mathbf{d}\hat{\mathbf{x}}}_{\mathbf{0}} + \boldsymbol{\xi}
\end{aligned}
\tag{27}
$$

where $\boldsymbol{\xi}$ denotes the second-order term in the Taylor series expansion of the adjusted observables for the perturbed free-net solution.

The disturbance of the adjusted observables, up to a second order, is:

$$\mathbf{d}\hat{\mathbf{y}} = \hat{\mathbf{y}}_{(\mathbf{H},\,\mathbf{c}+\mathbf{dc})} - \hat{\mathbf{y}}_{(\mathbf{H},\,\mathbf{c})} = \boldsymbol{\xi} \tag{28}$$

where each element of $\boldsymbol{\xi}$ is given by the quadratic expression

$$\xi_i = \frac{1}{2}\,\mathbf{d}\hat{\mathbf{x}}^{\mathrm{T}}\mathbf{Q}_i\,\mathbf{d}\hat{\mathbf{x}} \tag{29}$$

and $\mathbf{Q}_i$ is the Hessian matrix of the respective observation. Taking into account Eq. (20), the previous equation takes the form

$$\xi_i = \frac{1}{2}\,\mathbf{dc}^{\mathrm{T}}(\mathbf{E}\mathbf{H}^{\mathrm{T}})^{-1}\mathbf{E}\mathbf{Q}_i\mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{dc} \tag{30}$$

or equivalently

$$\xi_i = \frac{1}{2}\,\mathbf{d}\boldsymbol{\theta}^{\mathrm{T}}\mathbf{E}\mathbf{Q}_i\mathbf{E}^{\mathrm{T}}\mathbf{d}\boldsymbol{\theta} \tag{31}$$

Therefore, a MC disturbance causes a change in the adjusted observables which are implied by the original nonlinear model and thus affects the network's estimable characteristics. The last equation is particularly important as it relates the nonlinear variation in each adjusted observable to the perturbation of the non-estimable frame parameters.

The meaning of ξ_i

The terms ξ_i represent an important nonlinear element that has been neglected up to now in geodetic network analysis. Their values correspond to the geometrical effect of 'transformed linearization errors' between TRFs with respect to which a free-net solution can be determined. In our case, the corresponding frames arise from the MC systems $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$ and $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c} + \mathbf{dc}$, which do not necessarily share the same behavior regarding the influence of linearization errors in the free-net adjustment. From Eq. (30) we can conclude that the TRF stability matrix plays a role in controlling whether a MC disturbance is able to trigger significant linearization errors in the geometrical characteristics of the adjusted network.

Estimable TRF parameters

Following the notation given at the end of Sect. 3.3, let us model the difference between the initial ($\hat{\mathbf{x}}$) and the perturbed ($\hat{\mathbf{x}} + \mathbf{d}\hat{\mathbf{x}}$) MC solutions in terms of a (full) similarity transformation

$$
\begin{aligned}
\mathbf{d}\hat{\mathbf{x}} &= \mathbf{G}^{\mathrm{T}}\mathbf{dq} \\
&= \mathbf{E}^{\mathrm{T}}\mathbf{d}\boldsymbol{\theta} + \bar{\mathbf{E}}^{\mathrm{T}}\mathbf{d}\bar{\boldsymbol{\theta}}
\end{aligned}
\tag{32}
$$

where $\mathbf{d}\boldsymbol{\theta}$ and $\mathbf{d}\bar{\boldsymbol{\theta}}$ are the changes of the non-estimable and the estimable TRF parameters, respectively. Based on a simple LS adjustment of the above model and taking into account that the vector $\mathbf{d}\hat{\mathbf{x}}$ is given by the perturbation formula (20), we obtain the result

$$
\mathbf{d}\bar{\boldsymbol{\theta}} = \mathbf{0} \quad \text{and} \quad \mathbf{d}\boldsymbol{\theta} = (\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{dc}
\tag{33}
$$

as it should be expected due to the theoretical invariance of any estimable quantity under a MC perturbation within a free-net solution.

However, the aforementioned invariance is an apparent theoretical element that exists only within the linearized framework of the differential similarity transformation. A simple approach to quantify a likely variation of the estimable TRF characteristics in a **dc**-perturbed free network is to perform a stepwise LS estimation of the similarity transformation parameters from certain types of nonlinear datum functionals (the latter being respectively computed from the vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}} + \mathbf{d}\hat{\mathbf{x}}$). For example, a TRF scale-change factor may be directly estimated from chord differences over an independent set of network baselines, whereas TRF rota-

tion parameters can be obtained from the differences of appropriately formed directional angles among the network stations; for more details on such stepwise schemes for transformation parameter estimation see Leick and van Gelder (1975) and Han and van Gelder (2006). This approach has been actually implemented in a numerical example that is presented in Sect. 5.

4.3 A note on the TRF stability matrix

In the preceding sections we exposed the role of the matrix $(\mathbf{HE}^T)^{-1}$ in free-net adjustment problems. For any set of minimum constraints $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$ in a rank-deficient NEQ system, the aforementioned matrix dictates (i) the stability of the non-estimable frame parameters and (ii) the second-order nonlinear variation of the geometrical characteristics in the linearly adjusted network, *under the presence of a perturbation (error) in the MC vector* $\mathbf{c}$.

In every network adjustment there exists an algebraic form for the MCs such that their TRF stability matrix always becomes a unit matrix. Indeed, if we multiply the original system $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$ with the matrix $(\mathbf{HE}^T)^{-1}$ then an equivalent set of minimum constraints is obtained

$$(\mathbf{HE}^T)^{-1}\mathbf{H}(\mathbf{x} - \mathbf{x}_o) \ = \ (\mathbf{HE}^T)^{-1}\mathbf{c} \tag{34a}$$

or, in a more compact form

$$\mathbf{B}(\mathbf{x} - \mathbf{x}_o) \ = \ \mathbf{c}_\theta \tag{34b}$$

where $\mathbf{B} = (\mathbf{HE}^T)^{-1}\mathbf{H}$ and $\mathbf{c}_\theta = (\mathbf{HE}^T)^{-1}\mathbf{c}$. Obviously, the TRF stability matrix of the above MCs is

$$(\mathbf{BE}^T)^{-1} = ((\mathbf{HE}^T)^{-1}\mathbf{HE}^T)^{-1} = \mathbf{I} \tag{35}$$

whereas the perturbations of the non-estimable quantities in the adjusted network are now given by the simplified expressions

$$d\hat{\mathbf{x}} \ = \ \mathbf{E}^T d\mathbf{c}_\theta \tag{36}$$

$$d\boldsymbol{\theta} \ = \ d\mathbf{c}_\theta \tag{37}$$

Hence, it seems that the MCs in Eq. (34) are preferable for the implementation of a free-net adjustment as they ensure uniform stability and zero aliasing on the frame parameters under an error in the constrained datum functionals; see Eq. (21) vs. Eq. (37). This is however only a pseudo-regularization characteristic of the constrained LS adjustment since the effect of the original TRF stability matrix remains hidden within the vector $\mathbf{c}_\theta$ (and its possible variation) as indicated by the relationship

$$d\mathbf{c}_\theta \ = \ (\mathbf{HE}^T)^{-1}d\mathbf{c} \tag{38}$$

From a geodetic perspective, the importance of the matrix $(\mathbf{H}\mathbf{E}^T)^{-1}$ is due to the fact that a type-$\mathbf{dc}$ disturbance of a free network is more relevant than a generic type-$\mathbf{dc}_\theta$ disturbance. A brief explanation of this vital fact is provided in the rest of this section.

Despite the equivalency of the MC systems $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$ and $\mathbf{B}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}_\theta$ (i.e. both of them give theoretically the same free-net solution) their constrained elements are fundamentally different and they depend on some external TRF information ($\mathbf{x}^{\text{ext}}$) according to the hierarchical scheme:

$$\mathbf{x}^{\text{ext}} \quad \rightarrow \quad \boxed{\mathbf{c} = \mathbf{H}(\mathbf{x}^{\text{ext}} - \mathbf{x}_o)} \quad \rightarrow \quad \boxed{\mathbf{c}_\theta = (\mathbf{H}\mathbf{E}^T)^{-1}\mathbf{H}(\mathbf{x}^{\text{ext}} - \mathbf{x}_o)} \tag{39}$$

The vector $\mathbf{c}$ contains the reduced values for a minimum number of datum functionals, such as the coordinates at individual points, the azimuth of a specific baseline, or other more complicated types like the position/velocity of the network's centroid or the magnitude of the network's angular momentum over some or all of its stations. These values are determined within a linear approximation from $\mathbf{x}^{\text{ext}}$ and $\mathbf{x}_o$ – the MC matrix $\mathbf{H}$ contains the partial derivatives of the constrained datum functionals with respect to the network station positions. On the other hand, the vector $\mathbf{c}_\theta$ represents the (non-estimable) transformation parameters between $\mathbf{x}^{\text{ext}}$ and $\mathbf{x}_o$ that are inferred from the differences of the preceding datum functionals in each frame.

Both of the previous MC systems force the free-net solution $\hat{\mathbf{x}}$ to be computed in the same frame as $\mathbf{x}^{\text{ext}}$ in the following sense

$$\begin{aligned}
\boldsymbol{\theta}_{\hat{\mathbf{x}}, \mathbf{x}^{\text{ext}}} &= (\mathbf{H}\mathbf{E}^T)^{-1}\mathbf{H}(\hat{\mathbf{x}} - \mathbf{x}^{\text{ext}}) \\
&= (\mathbf{H}\mathbf{E}^T)^{-1}\mathbf{H}(\hat{\mathbf{x}} - \mathbf{x}_o) - (\mathbf{H}\mathbf{E}^T)^{-1}\underbrace{\mathbf{H}(\mathbf{x}^{\text{ext}} - \mathbf{x}_o)}_{\mathbf{c}} \\
&= (\mathbf{H}\mathbf{E}^T)^{-1}(\mathbf{H}(\hat{\mathbf{x}} - \mathbf{x}_o) - \mathbf{c}) \\
&= \mathbf{B}(\hat{\mathbf{x}} - \mathbf{x}_o) - \mathbf{c}_\theta \\
&= \mathbf{0}
\end{aligned} \tag{40}$$

that is, the (non-estimable) transformation parameters between $\hat{\mathbf{x}}$ and $\mathbf{x}^{\text{ext}}$ vanish when their determination is based on the datum functionals of the MC matrix $\mathbf{H}$.

Any hidden errors in the external TRF information imply a disturbance to the constrained elements in Eq. (39), thus causing a variation to the adjusted network as discussed in Sect. 4.1 and 4.2. The matrix $(\mathbf{H}\mathbf{E}^T)^{-1}$ is therefore the network's 'frame filter' against the $\mathbf{x}^{\text{ext}}$-related datum errors (which are given by

$\mathbf{dc} = \mathbf{H}\,\mathbf{dx}^{\mathrm{ext}}$) and it controls their propagated effect into the various components of the free-net solution.

5. Reference frame stability of MCs from a stochastic perspective

Thus far the role of the matrix $(\mathbf{HE}^{\mathrm{T}})^{-1}$ has been considered from a deterministic perspective in view of 'ill-conditioned' datum definition schemes in free networks. Such schemes arise from MCs with a problematic TRF stability matrix and they can lead to a mathematically correct but geodetically improper (frame unstable) free-net solution. Note that the frame-related instability does not interfere with the inversion of the constrained normal matrix $\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{WH}$ but it affects only the behavior of the matrix operator

$$(\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{WH})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W} \;=\; \mathbf{E}^{\mathrm{T}}(\mathbf{HE}^{\mathrm{T}})^{-1} \tag{41}$$

which acts on the constrained datum functionals $\mathbf{c}$ within the LS adjustment algorithm.

We shall now adopt a statistical view of Eqs. (20) and (21) so that we may evaluate the reference frame stability in a free-net solution through an appropriate covariance (CV) matrix $\boldsymbol{\Sigma}_\theta$. For this purpose, the perturbation vector $\mathbf{dc}$ in Eq. (21) is identified as a zero-mean random error which causes a corresponding zero-mean random error $\mathbf{d\theta}$ in the non-estimable TRF parameters of the adjusted network. By applying the error covariance propagation to Eq. (21), we get the formula:

$$\boldsymbol{\Sigma}_\theta \;=\; (\mathbf{HE}^{\mathrm{T}})^{-1}\boldsymbol{\Sigma}_c(\mathbf{EH}^{\mathrm{T}})^{-1} \tag{42}$$

where $\boldsymbol{\Sigma}_\theta$ and $\boldsymbol{\Sigma}_c$ denote the error CV matrices of the (non-estimable) TRF parameters and the constrained datum functionals, respectively. The latter is determined from the general equation

$$\boldsymbol{\Sigma}_c \;=\; \mathbf{H}\boldsymbol{\Sigma}_x^{\mathrm{prior}}\mathbf{H}^{\mathrm{T}} \tag{43}$$

which is obtained from Eq. (39) on the basis of a CV matrix $\boldsymbol{\Sigma}_x^{\mathrm{prior}}$ that specifies the accuracy of the external reference frame with respect to which the free-net solution is aligned. This matrix does not need to contain prior statistical information for all network stations but only for those involved in the underlying MCs, and it can be generally expressed as

$$\boldsymbol{\Sigma}_x^{\mathrm{prior}} \;=\; \begin{bmatrix} \boldsymbol{\Sigma}_{x_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

in accordance with an equivalent partition of the parameter vector $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^{\mathrm{T}} & \mathbf{x}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$,

where $\mathbf{x}_1$ refers to the network stations with known positions in the external reference frame and $\mathbf{x}_2$ refers to the remaining (new) network stations. In practice, it may originate either from the result of a previous adjustment, or by an empirical selection for the accuracy level of the available reference stations.

Taking into account Eqs. (42) and (43), we finally have the result

$$\mathbf{\Sigma}_\theta = (\mathbf{HE}^T)^{-1}\mathbf{H}\mathbf{\Sigma}_x^{\text{prior}}\mathbf{H}^T(\mathbf{EH}^T)^{-1} \tag{44}$$

which specifies, in a statistical sense, the TRF stability as a function of the adopted minimum constraints and the joint uncertainty (including the possible correlations) of the reference stations. The above matrix does not include the error contribution from the available measurements but it evaluates the frame stability that can be achieved by different choices of MCs in a given network.

The uncertainty of the estimated station positions due to the 'TRF-stability effect' is obtained by applying the error CV propagation to Eq. (20), thus yielding

$$\begin{aligned}
\bar{\mathbf{\Sigma}}_{\hat{x}} &= \mathbf{E}^T\mathbf{\Sigma}_\theta\mathbf{E} \\
&= \mathbf{E}^T(\mathbf{HE}^T)^{-1}\mathbf{\Sigma}_c(\mathbf{EH}^T)^{-1}\mathbf{E} \\
&= \mathbf{E}^T(\mathbf{HE}^T)^{-1}\mathbf{H}\mathbf{\Sigma}_x^{\text{prior}}\mathbf{H}^T(\mathbf{EH}^T)^{-1}\mathbf{E} \\
&= (\mathbf{I}-\mathbf{N}^-\mathbf{N})\mathbf{\Sigma}_x^{\text{prior}}(\mathbf{I}-\mathbf{N}^-\mathbf{N})^T
\end{aligned} \tag{45}$$

where the last equality stems from (11) and (12). The matrix $\bar{\mathbf{\Sigma}}_{\hat{x}}$ represents the contribution of the external frame's noise to the total accuracy of a free-net solution, and it is related to the following CV decomposition:

$$\mathbf{\Sigma}_{\hat{x}} = \mathbf{N}^-\mathbf{N}\mathbf{N}^- + \bar{\mathbf{\Sigma}}_{\hat{x}} \tag{46}$$

which is obtained when a full error propagation is applied to the generalized inversion formula in Eq. (7) under a stochastic interpretation for the auxiliary vector $\mathbf{z}$ (i.e. $\mathbf{\Sigma}_z \rightarrow \mathbf{\Sigma}_x^{\text{prior}}$).

The previous CV components, that is $\mathbf{N}^-\mathbf{N}\mathbf{N}^-$ and $\bar{\mathbf{\Sigma}}_{\hat{x}}$, correspond to $m{\times}m$ singular matrices with rank defect equal to m-rank$\mathbf{N}$ and rank$\mathbf{N}$ respectively, and they are both affected by the MC matrix $\mathbf{H}$. A well known result in free-net adjustment theory dictates that the inner constraints yield the best accuracy for the estimated positions in the sense that they minimize the trace of the first CV matrix in (46); see, e.g. Blaha (1982). Indeed, in this case (i.e. $\mathbf{H} = \mathbf{E}$) the reflexive generalized inverse $\mathbf{N}^-\mathbf{N}\mathbf{N}^-$ becomes equal to the pseudoinverse $\mathbf{N}^+$ which is known to have the smallest trace among all the symmetric reflexive generalized inverses of the NEQs matrix (Koch 1999, p. 62).

On the other hand, the trace minimization of the matrix $\bar{\mathbf{\Sigma}}_{\hat{x}}$ represents a zero-order

network optimization task which has not been unveiled in the geodetic literature (at least to the author's knowledge). The corresponding optimal $\mathbf{N}^-$ and its associated MC matrix $\mathbf{H}$ will provide the *most stable alignment of the free-net solution with an external frame that is characterized by an a priori CV matrix* $\mathbf{\Sigma}_x^{\text{prior}}$. The solution of such a problem does not always lead to the classic inner constraints ($\mathbf{H} = \mathbf{E}$) and its detailed treatment lies beyond the scope of the present paper.

6. Conclusions

The influence of the MCs on the reference frame stability in a free-net solution has been investigated in this paper. Our study considered the distortion effect due to a perturbation of the constrained datum functionals $\mathbf{c} = \mathbf{H}(\mathbf{x}^{\text{ext}} - \mathbf{x}_{\text{o}})$ by analyzing its propagation on the non-estimable and the estimable components of the adjusted network. The main findings along with some brief final remarks can be summarized as follows.

-   The matrix $(\mathbf{H}\mathbf{E}^{\text{T}})^{-1}$ plays a crucial role for the reference frame stability in a free-net solution, and it controls the impact of each MC to the (non-estimable) frame parameters of the adjusted network. This is a fundamental matrix which can be associated with alternative datum implementation strategies within the same physical network, but it could be also used for comparing the TRF stability from different network configurations with varying physical locations of their terrestrial stations.

-   A 'problematic' TRF stability matrix tends to amplify any external perturbation of the MC vector and also alias the positioning errors of the reference stations into different frame parameters.

-   Any errors that are present in the a priori positions of the reference stations cause a nonlinear distortion on the estimable characteristics of a free-net solution, and thus they affect the internal geometry of the minimum-constrained network (except for the case of a translation-only datum deficient network). The linearized LS framework which is used for the inversion of nonlinear rank-deficient models is 'blind' to this type of distortion, whose practical significance in geodetic applications remains to be investigated.

Based on the last of the previous findings, it is concluded that the datum choice problem may interfere with the determination of the geometrical form of a minimum-constrained network under a nonlinear observation model – the notion of a truly distortionless free-net solution requires therefore additional theoretical steps to be rigorously defined.

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