Potential field and smoothness conditions

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Abstract

The present study investigates the problem of band-limited information within gravity field. Due to a finite number of measurements at the surface or in satellite altitude a band-limitation is introduced both into the space domain and the frequency domain. Therefore, precaution is recommended to look for unbiasedly estimable functions of these bandlimited gravity field models. We propose the complementation of gravity field models by additional a priori information to obtain a complete model. The additional information is obtained by the smoothness of the potential field characterized by curvature conditions for the contour lines expressed by the first and second Beltrami operator.

1. Introduction

High spatial resolution models of the Earth's gravity field are derived from satellite missions such as GRACE and GOCE or from so-called combined model, where surface gravity and altimeter information is additional assimilated. These models open the door for various fields of geosciences to a deeper and more precise understanding of Earth processes. However, from a mathematical point of view the involved communities use different languages and, more importantly, different ways of representing the same information. While gravity fields are, on a global scale, usually represented by spherical harmonics, Earth process models often work with geographically gridded data. Therefore, the representation in the frequency domain in terms of spherical harmonics has to be transferred into a grid space domain representation. We follow the approach of *Losch et al.* (2002b, a) and expand the truncated model by priori information. The spherical harmonics as base functions allow us, due to the orthogonality relations, to split up the Hilbert space into sub-domains. With respect to the gravity field models we divide the space into three sub-domains: measurement, transition and omission domain. The first sub-domain is mainly fixed by the real measurements (e.g. satellite-to satellite tracking data, gravity gradient measurements, ...). In the transition zone the information of measurements are supported by the a priori knowledge about the smoothness of the estimated potential field. This additional information prevents the well-known over-estimation of information content for high frequencies. And finally the omission domain up to infinity is dominated only by the a priori knowledge about the smoothness of the potential field.

In this report we will recapitulate the mathematical background of the smoothness conditions, elaborate the behavior and compare the results with well known degree variance models (Kaula, 1966; Tscherning and Rapp, 1974) and up-to-date GRACE and GOCE models.

This paper is organized as follows. We will first summarize the special features of the spherical harmonics with regard to the notation in a Hilbert space HΓ (Sect. 2). The Parseval theorem gives us a direct connection between the norm in the Hilbert space and the degree variances. In sect. 3 we will introduce the first and second Beltrami differential operators, as indicators for the smoothness of vector fields by invariant scalar functions on a surface. We will specialize these operators for the sphere and show that the second Beltrami operator has the Laplace spherical harmonics as eigenfunctions (Sect. 4). With respect to the Sobolev conditions we will then be able to formulate bounds for the decrease of the degree variances with increasing degree of the spherical harmonics (Sect. 5). By means of Green's first formula we can extend this relation between the Beltrami operators and the spherical harmonics also for the first Beltrami operator. As the key result of sect. 5, we will obtain smoothness conditions in terms of upper bounds of the degree variances. At the end we will demonstrate the excellent agreement of the mathematical model with reality, especially with respect to the first analysis of real GOCE data by the time-wise approach (*Pail et al.*, 2010; *Schuh et al.*, 2010).

2. Potential field and spherical harmonics

The potential V in the exterior of the sphere satisfies the Laplace equation, which is given in Euclidian coordinates by

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{1}$$

and which can be rewritten in term of spherical coordinates with

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin\vartheta\cos\lambda \\ r\sin\sin\lambda \\ r\cos\vartheta \end{bmatrix} \tag{2}$$

(r radius vector, ϑ polar distance, λ geocentric longitude) in the form (Heiskanen and Moritz, 2000, eq. (1-41′))

$$\Delta V = r^2 \frac{\partial^2 V}{\partial r^2} + 2r\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial \vartheta^2} + \cot\vartheta \frac{\partial V}{\partial \vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2 V}{\partial \lambda^2} = 0\,. \tag{3}$$

The solution of the Laplace equation can be obtained by a separation ansatz (Heiskanen and Moritz (2000), p. 20-21) and results in the spherical harmonics. Every harmonic function outside the sphere $(r > 1)$ can be represented by

$$V(r,\vartheta,\lambda) = \sum_{\ell=0}^{\infty}\left(\frac{1}{r}\right)^{\ell+1}\sum_{m=0}^{\ell}\left[\bar{c}_{\ell m}\bar{P}_{\ell m}(\cos\vartheta)\cos m\lambda + \bar{s}_{\ell m}\bar{P}_{\ell m}(\cos\vartheta)\sin m\lambda\right]. \tag{4}$$

$\bar{P}_{\ell m}$ denotes the fully normalized Legendre polynomials depending on the harmonic degree ℓ and order m with the corresponding harmonic coefficients $\bar{c}_{\ell m}$ and $\bar{s}_{\ell m}$. The individual functions

$$\bar{C}_{\ell m}(\vartheta,\lambda) := \bar{P}_{\ell m}(\cos\vartheta)\cos m\lambda \tag{5}$$

$$\bar{S}_{\ell m}(\vartheta,\lambda) := \bar{P}_{\ell m}(\cos\vartheta)\sin m\lambda \tag{6}$$

forms a complete set of orthonormal base functions on the unit sphere Γ. Their behavior can be summarized by

$$\iint_{\Gamma}\left[\bar{C}_{\ell m}(\vartheta,\lambda)\right]^2 d\sigma = \iint_{\Gamma}\left[\bar{S}_{\ell m}(\vartheta,\lambda)\right]^2 d\sigma = 4\pi$$

$$\left.\begin{aligned}&\iint_{\Gamma}\bar{C}_{\ell m}(\vartheta,\lambda)\,\bar{C}_{sr}(\vartheta,\lambda)d\sigma = 0\\ &\iint_{\Gamma}\bar{S}_{\ell m}(\vartheta,\lambda)\,\bar{S}_{sr}(\vartheta,\lambda)d\sigma = 0\end{aligned}\right\}\quad \text{if } s \neq \ell \text{ or } r \neq m \text{ or both} \tag{7}$$

$$\iint_{\Gamma}\bar{C}_{\ell m}(\vartheta,\lambda)\,\bar{S}_{\ell m}(\vartheta,\lambda)d\sigma = 0 \quad \text{in any case}$$

where we use the abbreviations $\iint_{\Gamma} := \int_{\lambda=0}^{2\pi}\int_{\vartheta=0}^{\pi}$ for the integral over the unit sphere Γ and $d\sigma := \sin\vartheta\, d\vartheta\, d\lambda$ for the surface element on the unit sphere.

2.1 Hilbert space of functions on the unit sphere Γ

Let $u(\vartheta,\lambda)$ and $v(\vartheta,\lambda)$ be functions of the inner product space $\mathcal{L}_{\Gamma}^{2}$ on the unit sphere Γ. The inner product is defined by

$$\left\langle u(\vartheta,\lambda), v(\vartheta,\lambda)\right\rangle_{\mathcal{H}_{\Gamma}} := \iint_{\Gamma} u(\vartheta,\lambda)\, v(\vartheta,\lambda) d\sigma \tag{8}$$

and the norm by

$$\left\|u(\vartheta,\lambda)\right\|_{\mathcal{H}_\Gamma} := \left\{\iint_\Gamma \left[u(\vartheta,\lambda)\right]^2 d\sigma\right\}^{1/2}. \tag{9}$$

We assume that the function is square integrable, that is,

$$\iint_\Gamma \left[u(\vartheta,\lambda)\right]^2 d\sigma < \infty \tag{10}$$

As mentioned already in the last section the functions $\overline{C}_{\ell m}(\vartheta,\lambda)$ and $\overline{S}_{\ell m}(\vartheta,\lambda)$ form a complete orthogonal basis. A complete inner product space is called a Hilbert space $\mathcal{H}_\Gamma$ (Meissl (1975), p. 36-38), and shortly characterized by

$$\mathcal{H}_\Gamma := \{u:(\vartheta,\lambda)\to\mathbb{R} \,|\, u\in\mathcal{L}_2\}. \tag{11}$$

Because of the completeness of the basis any function $u(\vartheta,\lambda)$ on the sphere can be represented by the base functions $\overline{C}_{\ell m}(\vartheta,\lambda)$ and $\overline{S}_{\ell m}(\vartheta,\lambda)$ in the form

$$u(\vartheta,\lambda) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left[\overline{c}_{\ell m}\overline{C}_{\ell m}(\vartheta,\lambda) + \overline{s}_{\ell m}\overline{S}_{\ell m}(\vartheta,\lambda)\right]. \tag{12}$$

2.2 Parseval equation for spherical harmonics

The application of orthogonality relations (7) allows us now to reformulate the norm (9) of an arbitrary function $u(\vartheta,\lambda)$ on the sphere Γ. If we introduce (12) in (9) we get

$$\left\|u(\vartheta,\lambda)\right\|_{\mathcal{H}_\Gamma}^2 = \iint_\Gamma \left[\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left[\overline{c}_{\ell m}\overline{C}_{\ell m}(\vartheta,\lambda) + \overline{s}_{\ell m}\overline{S}_{\ell m}(\vartheta,\lambda)\right]\right]^2 d\sigma. \tag{13}$$

Solving the quadratic form, interchanging the sums with the integrals and considering the orthogonality relations (7) all mixed terms vanish and we obtain

$$\left\|u(\vartheta,\lambda)\right\|_{\mathcal{H}_\Gamma}^2 = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\left[\overline{c}_{\ell m}^2 + \overline{s}_{\ell m}^2\right], \tag{14}$$

which is denoted as *Parseval equation*. Often the summation over the order m is summarized in *degree variances*

$$\sigma_\ell^2 := \sum_{m=0}^{\ell}\left[\overline{c}_{\ell m}^2 + \overline{s}_{\ell m}^2\right], \tag{15}$$

Thus, the Parseval equation can be written in compact form as

$$\|u(\vartheta,\lambda)\|^2_{\mathcal{H}} = 4\pi \sum_{\ell=0}^{\infty} \sigma_\ell^2 \,, \tag{16}$$

3. Beltrami differential operators on the sphere

Let $u(\vartheta,\lambda)=const$ be an equi-potential line on the continuous surface Φ. The coefficients of Gauss's first fundamental form

$$ds^2 = Ed\vartheta^2 + 2Fd\vartheta d\lambda + Gd\lambda^2 \tag{17}$$

with respect to position vector $\boldsymbol{x}(\vartheta,\lambda)$ and the surface parameters ϑ and λ are defined by (Kreyszig, 1993, Sec. 9.6, p. 543)

$$\begin{aligned}
E &= (\boldsymbol{x}_\vartheta,\boldsymbol{x}_\vartheta) = \left(\frac{\partial x}{\partial \vartheta}\right)^2 + \left(\frac{\partial y}{\partial \vartheta}\right)^2 + \left(\frac{\partial z}{\partial \vartheta}\right)^2 := x_\vartheta^2 + y_\vartheta^2 + z_\vartheta^2 \\
F &= (\boldsymbol{x}_\vartheta,\boldsymbol{x}_\lambda) = \frac{\partial x}{\partial \vartheta}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \vartheta}\frac{\partial y}{\partial \lambda} + \frac{\partial z}{\partial \vartheta}\frac{\partial z}{\partial \lambda} := x_\vartheta x_\lambda + y_\vartheta y_\lambda + z_\vartheta z_\lambda \\
G &= (\boldsymbol{x}_\lambda,\boldsymbol{x}_\lambda) = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2 := x_\lambda^2 + y_\lambda^2 + z_\lambda^2
\end{aligned} \tag{18}$$

with $\boldsymbol{x}_\vartheta = \dfrac{\partial \boldsymbol{x}}{\partial \vartheta}(\vartheta,\lambda)$ and where $(\cdot,\cdot)$ denote the scalar product of vectors. x_ϑ stands for $\dfrac{\partial x}{\partial \vartheta}(\vartheta,\lambda)$ and similarly we will use u_ϑ as an abbreviation for $\dfrac{\partial u}{\partial \vartheta}(\vartheta,\lambda)$. The same conventions are also introduced for other parameters. The square of the length of the normal vector is defined by

$$W^2 = EG - F^2 \,. \tag{19}$$

Let $u(\vartheta,\lambda)$ be a differentiable function on Φ, then the tangent vector is given by

$$\boldsymbol{t}_u = \frac{u_\lambda \boldsymbol{x}_\vartheta - u_\vartheta \boldsymbol{x}_\lambda}{W} \,. \tag{20}$$

and the square of the norm of this vector

$$|\boldsymbol{t}_u|^2 = (\boldsymbol{t}_u,\boldsymbol{t}_u) = \frac{Eu_\lambda - 2Fu_\vartheta u_\lambda + Gu_\lambda}{W^2} := \nabla_\Phi u \,. \tag{21}$$

defines *Beltrami's first differential operator* $\nabla_\Phi u$, which is a parameter invariant scalar function on the surface Φ (Strubecker (1958), p. 90). This quantity is connected with the gradient vector $\mathbf{grad}_\Phi u$ through

$$\mathbf{grad}_\Phi u = \frac{(Gu_\vartheta - Fu_\lambda)\boldsymbol{x}_\vartheta + (Eu_\lambda - Fu_\vartheta)\boldsymbol{x}_\vartheta}{W^2}, \tag{22}$$

which defines a vector tangential to the surface Φ, pointing in the direction of the deepest descent, orthogonal to the equi-potential line. The squared length of the gradient vector is equivalent to Beltrami's first differential operator applied to u

$$\left|\mathbf{grad}_\Phi u\right|^2 = \left(\mathbf{grad}_\Phi u,\ \mathbf{grad}_\Phi u\right) = \nabla_\Phi u\,. \tag{23}$$

If the function $u(\vartheta,\lambda)$ is twice continuously differentiable then

$$\Delta_\Phi u := \mathrm{div}\mathbf{grad}_\Phi u = \frac{1}{W}\left[\frac{\partial}{\partial\vartheta}\left(\frac{Gu_\vartheta - Fu_\lambda}{W}\right) + \frac{\partial}{\partial\lambda}\left(\frac{Eu_\lambda - Eu_\vartheta}{W}\right)\right] \tag{24}$$

defines *Beltrami's second differential operator* $\Delta_\Phi u$, commonly denoted as *Laplace-Beltrami-operator*. Beltrami's differential operators are often discussed in connection with Green's first formula (Kreyszig, 1993, Sec. 9.8, p. 553)

$$\iint_B (\mathbf{grad}_\Phi u, \mathbf{grad}_\Phi v)\, d\Phi = \oint_{\partial B} u(\mathbf{grad}_\Phi v, \boldsymbol{\nu})\, d\beta - \iint_B u\Delta_\Phi v\, d\Phi \tag{25}$$

for two equi-potential lines $u(\vartheta,\lambda)$ and $v(\vartheta,\lambda)$. Here B denotes a subarea on the surface Φ and ∂B marks its boundary. Let β be the arc length along the boundary and let the unit vector $\boldsymbol{\nu}$ be tangential to the surface Φ and normal outward to B. If we consider the case of the sphere Γ this formula becomes

$$\iint_\Gamma (\mathbf{grad}_\Gamma u, \mathbf{grad}_\Gamma v)\, d\Gamma = -\iint_\Gamma u\Delta_\Gamma v\, d\Gamma \tag{26}$$

(Meissl (1971), p. 12). A further specialization leads to

$$\iint_\Gamma (\mathbf{grad}_\Gamma u, \mathbf{grad}_\Gamma v)\, d\Gamma = \iint_\Gamma \nabla_\Gamma u\, d\Gamma = -\iint_\Gamma u\Delta_\Gamma u\, d\Gamma\,, \tag{27}$$

and gives a connection between Beltrami's first and second differential operator on the sphere. Introducing Gauss's first fundamental coefficients for the sphere Γ, with $r = R = const.$, we get

$$E = R^2,\quad F = 0,\quad G = R^2 \sin 2\vartheta \quad \text{and} \quad W = R^2 \sin\vartheta\,. \tag{28}$$

Beltrami's first differential operator (21) on the sphere Γ is defined by

$$\nabla_\Gamma u = \frac{1}{R^2}\left(\frac{\partial u}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial u}{\partial\lambda}\right) \tag{29}$$

and Beltrami's second differential operator (24) can be rewritten as

$$\Delta_\Gamma u = \frac{1}{R^2 \sin\vartheta}\left(\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right)+\frac{\partial^2 u}{\partial\lambda^2}\right). \tag{30}$$

After differentiation we get

$$\Delta_\Gamma u = \frac{1}{R^2}\frac{\partial^2 u}{\partial\vartheta^2}+\frac{\cot\vartheta}{R^2}\frac{\partial u}{\partial\vartheta}+\frac{1}{R^2\sin^2\vartheta}\frac{\partial^2 u}{\partial\lambda^2}. \tag{31}$$

4. Laplace-Beltrami operator and spherical harmonics

Starting from the Laplace equation in spherical form (3) we follow the separation approach (Heiskanen and Moritz (2000), sec. 1-9) and split up the function $V(r, \vartheta, \lambda)$ into a radial part $f(r)$ and into an angular part $Y(\vartheta, \lambda)$

$$V(r,\vartheta,\lambda) = f(r)\,Y(\vartheta,\lambda) \tag{32}$$

This separation ansatz allows us now to reformulate the partial derivatives of (3) and we end up with

$$\begin{aligned}\Delta V(r,\vartheta,\lambda) &= r^2 Y(\vartheta,\lambda)\,f''(r)+2rY(\vartheta,\lambda)\,f'(r)+\\ &+f(r)\left(\frac{\partial^2 Y}{\partial\vartheta^2}(\vartheta,\lambda)+\cot\vartheta\frac{\partial Y}{\partial\vartheta}(\vartheta,\lambda)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\lambda^2}(\vartheta,\lambda)\right),\end{aligned} \tag{33}$$

where the primes denote differentiation with respect to the argument r. Using the Laplace condition $\Delta V(r,\vartheta,\lambda)=0$, multiplying (33) by $f(r)\,Y(\vartheta,\lambda)=0$ allows for a separation into the radial dependent part on the left side and into the angular dependent part on the right side

$$\begin{aligned}&\frac{1}{f(r)}\left((r^2 f''(r)+2rf'(r)\right)=\\ &=-\frac{1}{Y(\vartheta,\lambda)}\left(\frac{\partial^2 Y}{\partial\vartheta^2}(\vartheta,\lambda)+\cot\vartheta\frac{\partial Y}{\partial\vartheta}(\vartheta,\lambda)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\lambda^2}(\vartheta,\lambda)\right).\end{aligned} \tag{34}$$

This differential equation can only be satisfied if both sides are constant. Choosing this constant by $\ell(\ell-1)$ we obtain for the left hand side the Euler-Cauchy equation (Kreyszig, 1993, Sec. 2.6, p. 73)

$$r^2 f_\ell''(r)+2rf_\ell'(r)-\ell(\ell-1)f_\ell(r)=0 \tag{35}$$

and for the right hand side

$$\frac{\partial^2 Y_\ell}{\partial \vartheta^2}(\vartheta,\lambda)+\cot\vartheta\frac{\partial Y_\ell}{\partial \vartheta}(\vartheta,\lambda)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 Y_\ell}{\partial \lambda^2}(\vartheta,\lambda)+\ell(\ell-1)\,Y_\ell(\vartheta,\lambda)=0\,. \quad (36)$$

The first three terms of this second order differential equation (36) can be rewritten in terms of Beltrami's second differential operator (31) as

$$R^2\Delta_\Gamma Y_\ell(\vartheta,\lambda)=\frac{\partial^2 Y_\ell}{\partial \vartheta^2}(\vartheta,\lambda)+\cot\vartheta\frac{\partial Y_\ell}{\partial \vartheta}(\vartheta,\lambda)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 Y_\ell}{\partial \lambda^2}(\vartheta,\lambda)\,. \quad (37)$$

Using this abbreviation, equation (36) becomes

$$\Delta_\Gamma Y_\ell(\vartheta,\lambda)=-\frac{1}{R^2}\ell(\ell+1)\,Y_\ell(\vartheta,\lambda) \quad (38)$$

and shows that the Laplace-Beltrami operator has as eigenfunctions the Laplace surface spherical harmonics with the eigenvalues

$$\lambda_\ell=-\frac{1}{R^2}\ell(\ell+1) \quad (39)$$

(cf. Helfrich (2007), p. 46). In contrast to Laplace's equation $\Delta V=0$ (cf. eq. (1) or (3)), which defines the behavior outside the sphere, the Laplace-Beltrami operator $\Delta_\Gamma Y_\ell(\vartheta,\lambda)$ characterizes the behavior of V on the surface of the sphere Γ.

5. Smoothness of a potential field

Let $u(\vartheta,\lambda)$ be a continuous function on the sphere Γ, which is twice continuously differentiable at least in the weak sense, that is the derivation $v(s)$ of the function $u(s)$ with $s=s(\vartheta,\lambda)$ satisfies the Sobolev-condition

$$\int_a^b u(s)\,\varphi'(s)ds=-\int_a^b v(s)\,\varphi(s)ds \quad (40)$$

for every continuous function $\varphi(s)\in C(\Gamma)$ on the sphere Γ. The metric is defined by the inner product (8) and the norm by (9), respectively. Let us assume that the function $u(\vartheta,\lambda)$ is square integrable, that is,

$$\|u(\vartheta,\lambda)\|^2_{\mathcal{H}_\Gamma}=\iint_\Gamma [u(\vartheta,\lambda)]^2\,d\sigma<\infty\,, \quad (41)$$

which means that the function $u(\vartheta,\lambda)$ is defined in the Hilbert space $\mathcal{H}_\Gamma$. As elaborated in Sect. 2.2 the Parseval equation

$$\|u(\vartheta,\lambda)\|^2_{\mathcal{H}_\Gamma} = 4\pi\sum_{\ell=0}^{\infty}\sigma_\ell^2 \tag{16}$$

gives us a direct connection between the norm of the function and the degree variances. A finite norm immediately implies that the sum of the degree variances converges to a finite value, i.e.

$$4\pi\sum_{\ell=0}^{\infty}\sigma_\ell^2 < \infty\,. \tag{42}$$

It is well known that the series $\sum_{\ell=0}^{\infty}\frac{1}{\ell}$ diverges, but $\sum_{\ell=0}^{\infty}\frac{1}{\ell^{1+\varepsilon}}$ converges for $\varepsilon > 0$.

Therefore, the condition (42) is equvalent to

$$\sigma_\ell^2 < \frac{c}{\ell}\,. \tag{43}$$

and gives a requirement for the degree variances with respect to a finite norm $\|u(\vartheta,\lambda)\|^2_{\mathcal{H}_\Gamma}$. To get a smoother function we can define further restrictions on the function. In general the Hilbert space $\mathcal{H}_\Gamma^1$ for function with existing (weak) derivatives is defined by the inner product

$$\langle u(\vartheta,\lambda),\, v(\vartheta,\lambda)\rangle_{\mathcal{H}_\Gamma^1} := \langle u(\vartheta,\lambda),\, v(\vartheta,\lambda)\rangle_{\mathcal{H}_\Gamma} + \langle Du(\vartheta,\lambda),\, Dv(\vartheta,\lambda)\rangle_{\mathcal{H}_\Gamma} \tag{44}$$

and the norm

$$\|u(\vartheta,\lambda)\|_{\mathcal{H}_\Gamma^1} = \|u(\vartheta,\lambda)\|_{\mathcal{H}_\Gamma} + \|Du(\vartheta,\lambda)\|_{\mathcal{H}_\Gamma} \tag{45}$$

where D defines a differential operator for the first derivative. The same can be done for arbitrary highly derivatives. This way, the spaces become smoother and smoother if we define the inner product by

$$\langle u(\vartheta,\lambda),\, v(\vartheta,\lambda)\rangle_{\mathcal{H}_\Gamma^p} = \sum_{k=0}^{p}\left(\langle D^{(k)}u(\vartheta,\lambda),\, D^{(k)}v(\vartheta,\lambda)\rangle_{\mathcal{H}_\Gamma}\right) \tag{46}$$

and the norm by

$$\|u(\vartheta,\lambda)\|_{\mathcal{H}_\Gamma^p} = \sum_{k=0}^{p}\left(\|D^{(k)}u(\vartheta,\lambda)_{\mathcal{H}_\Gamma}\|\right) \tag{47}$$

where $D^{(k)}$ defines a differential operator for the k^{th} derivative.

In the following we will focus our attention to the Hilbert spaces $\mathcal{H}_\Gamma^1$ and $\mathcal{H}_\Gamma^2$.

Let us start with an example of the Hilbert space $\mathcal{H}_\Gamma^2$ on the sphere. Here, the norm of the second derivative and therefore the norm of Beltrami's second differential operator elaborated in Sect. 3 is of special interest. Recalling the particular property of Beltrami's second differential operator with respect to the surface spherical harmonics as eigenfunctions (cf. Sect. 4) we are able to compute the norm of the second derivatives, which are responsible for the smoothness of the functions and dominant over all other terms in the sum of (47).

Taking the norm of the Laplace-Beltrami operator (31) and using (38) we get

$$\left\|\Delta_\Gamma Y(\vartheta,\lambda)\right\|_{\mathcal{H}_\Gamma}^2 = \left\|\sum_{\ell=0}^{\infty}\Delta_\Gamma Y_\ell(\vartheta,\lambda)\right\|_{\mathcal{H}_\Gamma}^2 = \left\|\sum_{\ell=0}^{\infty}-\frac{1}{R^2}\ell(\ell+1)Y_\ell(\vartheta,\lambda)\right\|_{\mathcal{H}_\Gamma}^2 =$$

$$= \iint_\Gamma \left[\sum_{\ell=0}^{\infty}-\frac{1}{R^2}\ell(\ell+1)Y_\ell(\vartheta,\lambda)\right]^2 d\sigma \tag{48}$$

Interchanging summation and integration and recalling that all mixed terms vanish due to the orthogonality relations (7) we obtain

$$\left\|\Delta_\Gamma Y(\vartheta,\lambda)\right\|_{\mathcal{H}_\Gamma}^2 = \sum_{\ell=0}^{\infty}\left(\frac{1}{R^2}\ell(\ell+1)\right)^2 \iint_\Gamma \left[Y_\ell(\vartheta,\lambda)\right]^2 d\sigma =$$

$$= \frac{4\pi}{R^4}\sum_{\ell=0}^{\infty}-\frac{1}{R^2}\left(\ell(\ell+1)\right)^2\sigma_\ell^2 \,. \tag{49}$$

Requiring a finite norm of the Laplace-Beltrami operator

$$\left\|\Delta_\Gamma Y(\vartheta,\lambda)\right\|^2 < \infty \tag{50}$$

is equivalent to the constraint

$$\sum_{\ell=0}^{\infty}\ell^4\sigma_\ell^2 < \infty \tag{51}$$

and means that the series must converge. We employ the same argument as before regarding the convergence of an infinite series $\sum_{\ell=0}^{\infty}\frac{1}{\ell^{1+\varepsilon}}$ with $\varepsilon > 0$ (cf. (43)). This yields the condition

$$\ell^4\sigma_\ell^2 < \frac{c}{\ell} \tag{52}$$

for the behavior of the degree variances σ_ℓ^2, if we look for functions in the Hilbert space $\mathcal{H}_\Gamma^2$ with squared integrable second derivative.

Green's first formula specialized for the whole sphere (28) opens up the possibility to interpret the mixed inner product

$$\iint_\Gamma Y(\vartheta,\lambda)\Delta_\Gamma Y(\vartheta,\lambda)d\Gamma = -\iint_\Gamma \big(\mathbf{grad}_\Gamma Y(\vartheta,\lambda),\ \mathbf{grad}_\Gamma Y(\vartheta,\lambda)\big)d\Gamma \tag{53}$$

as an extended norm with respect to the scalar product of the tangent vector $\mathbf{grad}_\Gamma u$. As already stated in (28) Green's first formula gives the connection between Beltrami's first and second differential operator and the squared length of the tangential vectors $\boldsymbol{t}_u$ and $\mathbf{grad}_\Gamma u$, respectively. This allows us to reformulate the inner product (53) as a condition which guarantees the convergence of the first derivatives of the function $u(\vartheta,\lambda)$ on the sphere Γ. Again we use the direct connection between Beltrami's second differential operator and the eigenfunctions (38) to write

$$\iint_\Gamma Y(\vartheta,\lambda)\Delta_\Gamma Y(\vartheta,\lambda)d\Gamma = -\iint_\Gamma \sum_{\ell=0}^{\infty} -\frac{1}{R^2}\,\ell(\ell+1)\,Y_\ell(\vartheta,\lambda)\,Y_\ell(\vartheta,\lambda)d\Gamma\,. \tag{54}$$

Interchanging summation and integration, and using the orthogonality relations (7) with respect to the Parseval theorem (16) yields

$$\iint_\Gamma Y(\vartheta,\lambda)\Delta_\Gamma Y(\vartheta,\lambda)d\Gamma = \frac{4\pi}{R^2}\sum_{\ell=0}^{\infty}\ell(\ell+1)\,\sigma_\ell^2\,. \tag{55}$$

Combining (53) and (55) we get

$$\iint_\Gamma \big(\mathbf{grad}_\Gamma Y(\vartheta,\lambda),\ \mathbf{grad}_\Gamma Y(\vartheta,\lambda)\big)d\Gamma = -\iint_\Gamma \nabla_\Gamma Y(\vartheta,\lambda)d\Gamma = \frac{4\pi}{R^2}\sum_{\ell=0}^{\infty}\ell(\ell+1)\sigma_\ell^2 \tag{56}$$

note the close connection between the squared length of the gradient vector i.e. of Beltrami's first differential operator, and the degree variances. The condition for a finite norm of Beltrami's first operator on the sphere reads

$$\ell^2\sigma_\ell^2 < \frac{c}{\ell} \;\Rightarrow\; \sigma_\ell^2 < \frac{c}{\ell^3}\,. \tag{57}$$

Table 1 provides a compact summary of the key results of this section in which we have elaborated the relationship between smoothness of a function $u(\vartheta,\lambda)$, represented by the base functions $Y(\vartheta,\lambda)$ on the sphere Γ, and the degree variances σ_ℓ^2.

Table 1: *Relation between Hilbert space, smoothness and size of the degree variances*

Hilbert space	*smoothness*	*constraints*	*degree variances*
$\|u(\vartheta,\lambda)\|_{\mathcal{H}_\Gamma}$	function	$\iint_\Gamma (Y(\vartheta,\lambda))^2\, d\Gamma < \infty$	$\sigma_\ell^2 < \dfrac{c}{\ell^1}$
$\|u(\vartheta,\lambda)\|_{\mathcal{H}_\Gamma^1}$	1st derivative	$\iint_\Gamma \left\|\mathbf{grad}_\Gamma (Y(\vartheta,\lambda))^2\, d\Gamma\right\| < \infty$	$\sigma_\ell^2 < \dfrac{c}{\ell^3}$
$\|u(\vartheta,\lambda)\|_{\mathcal{H}_\Gamma^2}$	2st derivative	$\iint_\Gamma (\Delta_\Gamma Y(\vartheta,\lambda))^2\, d\Gamma < \infty$	$\sigma_\ell^2 < \dfrac{c}{\ell^5}$

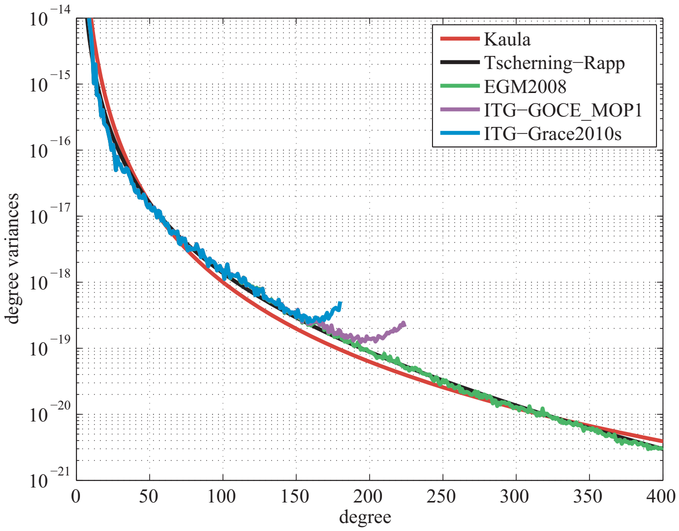

Fig. 1: *Comparison of degree variances from model computations (Kaula, Tschening-Rapp) and up-to-date gravity field models from measurements ITG-GRACE2010s and EGM2008 (ICGEM, 2010) in comparison with an unconstrained (only polar gap regularization) internal version of the ESA-GOCE-HPF model GO CONS EGM TIM 2i (Pail et al., 2010).*

6. Benefit of the smoothness condition and conclusions

Prior information about the smoothness of the potential field, obtained for instance from the comparison with measured surface or satellite data, allows us to restrict the mathematical model to satisfy certain smoothness conditions. For the actual behavior of the Earth's gravity field many studies confirm *Kaula's rule of thumb*

$$\frac{\sigma_\ell}{\sqrt{2\ell+1}} \sim \pm\frac{10^{-5}}{\ell^2} \quad \Rightarrow \quad \sigma_\ell^2 \sim \frac{10^{-10}(2\ell+1)}{\ell^4} \tag{58}$$

which was elaborated in an extensive study of gravity anomalies (Kaula (1966),
1
p. 98). Studies performed by Tscherning and Rapp (1974) confirm the $1/\ell^3$ behavior of the degree variances given by

$$\sigma_\ell^2 = \frac{R^2 A}{(\ell-1)(\ell-2)(\ell-B)} \tag{59}$$

where R denotes the radius of the Bjerhammar sphere, and A and B are empirical constants determined from gravity anomaly data.

Figure 1 summarizes the degree variances of up-to-date global gravity fields (cf. e.g. ICGEM (2010)) derived from GRACE and combined models, where additional data like surface gravity data and satellite altimetry data are assimilated (ICGEM, 2010) and GOCE models (*Pail et al*., 2010). A comparison with the smoothness conditions in Tab. 1 represented by Kaula's rule shows now, that the behavior of the Earth's gravity field corresponds to the smoothness conditions of the Hilbert space $\mathcal{H}_\Gamma^1$, where the norm of the first derivative is finite.

Smoothness considerations can be utilized in the modeling process in different ways. As deterministic approach by introducing hybrid norms (Tikhonov and Arsenin, 1977, Sect. III,p. 95) or restrictions to the upper limit of the parameters (Roese-Koerner, 2009) and as stochastic approaches by collocation (Moritz, 1980) and constructions of tailored covariance functions (*Arabelos et al*., 2007).

Dedication: To Dimitrios Arabelos, with whom I had the pleasure of collaborating in some projects (GEOMED, MANICORAL, CIGAR, E2mGal) and to discuss many topics, especially with respect to the GOCE mission and the design of tailored covariance functions for the collocation approach. Thank you for these precious and fruitful discussions and your extraordinary research impacts.

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