Datum Singularity in TRF Estimation: Diagnostic Tools and Rank-Deficient NEQ Reconstruction

C. Kotsakis

Department of Geodesy and Surveying, Aristotle University of Thessaloniki, University Box 440, Thessaloniki 54124, Greece

Abstract: The datum-related singularity of the input normal equations (NEQ) is a crucial element in the context of terrestrial reference frame (TRF) estimation under the minimal-constraint framework. However, this element is often missing in the recovered NEQ from SINEX files after the usual de-constraining based on the stated information for the stored solutions. The same setback also occurs with the original NEQ that are formed by the least-squares processing of space geodetic data due to datum information which is carried by various modeling choices or other software-dependent procedures. In the absence of this prior singularity, it is not possible to obtain genuine minimally-constrained solutions because of the interference between the input NEQ's content and the external datum conditions, a fact that may alter the geometrical information of the original measurements and can cause unwanted distortions in the estimated solution. The goal of this study is the formulation of a filtering scheme to enforce the proper singularity in the input NEQ with regard to datum parameters that will be handled by the minimal-constraint setting in TRF estimation problems. The importance of this task is extensively discussed and justified with the help of several numerical examples in different GNSS networks.

Keywords: NEQ, rank defect, minimal constraints, TRF estimation, frame distortion.

1. Introduction

The least-squares estimation of station positions in geodetic networks, and the akin task of terrestrial reference frame (TRF) realization, is a rank-deficient problem without a unique stable solution. This is a well known aspect which reflects the inability of geodetic measurements to define by themselves, and at sufficient accuracy level, all the required components of a terrestrial coordinate system (Sillard and Boucher 2001; Dermanis 2003). Typically, the solution to such problem is derived by the constrained inversion of a system of normal equations (NEQ) using an appropriate set of external datum conditions to complement the lack of data information. What is more, the rank defect of this problem is a prerequisite for applying (and justifying the use of) the *minimal constraints theory* in support of the afore-

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said NEQ inversion. This operational approach is widely used for the primary realization of TRFs, as well as in the alignment of geodetic networks to already existing frames, and it is known to impose important and desirable properties in the estimated solution (Altamimi et al. 2002; Angermann et al. 2004; Altamimi and Dermanis 2009; Legrand et al. 2010; Kotsakis 2013; Glaser et al. 2015). However, the minimal constraints are frequently employed in practice without verifying the true rank defect of the input NEQ at hand, and thus their role should often be accepted rather loosely (e.g. without any guarantee for the distortion-free property of the adjusted network) by evoking the notion of the "practically-minimal" datum conditions that was introduced by Sillard and Boucher (2001).

For various reasons the aforementioned rank defect does not always occur in strict numerical sense, a fact that may cause unwanted effects in "minimally-constrained" solutions from space geodetic data. If the input NEQ are non-singular then any set of additive datum constraints will generally affect the geometrical characteristics of the estimated solution, and it will lead to a distorted network in the sense of altering the information content of the used measurements. This, in turn, may affect other inferences of scientific interest such as the quantification of nonlinear signals (and their geophysical interpretation) in a time series of "minimally-constrained" solutions with respect to a global secular frame, or the statistical testing on well-estimable quantities by producing false judgments with type I or type II errors. In our view this problem has not been sufficiently addressed in the geodetic literature and it is often overlooked with regard to its practical relevance for TRF applications. A general diagram that outlines the implications due to non-singularity in the input NEQ under the minimal-constraint estimation framework is given in Fig. 1.

Apart from being a primary constituent of network adjustment theory, the NEQ rank defect is also an essential apparatus that can facilitate (i) the purification of the datum definition process from the data noise and other observation/technique-related limitations, and (ii) the independence of the estimated geometrical characteristics in a geodetic network from the external datum conditions. What we imply here is the desire to work with singular NEQ whose rank defect is *deliberately* tuned to datum parameters that we intend to fix by minimal constraints on the basis of high-quality reference stations. This not only will assure that the datum definition does not interfere with geodetically estimable elements like the network geometry and its temporal variation, but also that the datum definition itself remains unaffected by the used observations and their associated errors. Such de-coupling requires the conversion of the input NEQ to a strictly singular system with equivalent geometrical content and rank defect linked to pre-selected datum parameters.

Following the previous discussion, the problem that we will study herein is the swapping of a normal system from a (weakly) full-rank form to a (strictly) rank-deficient form by removing its internal information for a prescribed set of datum

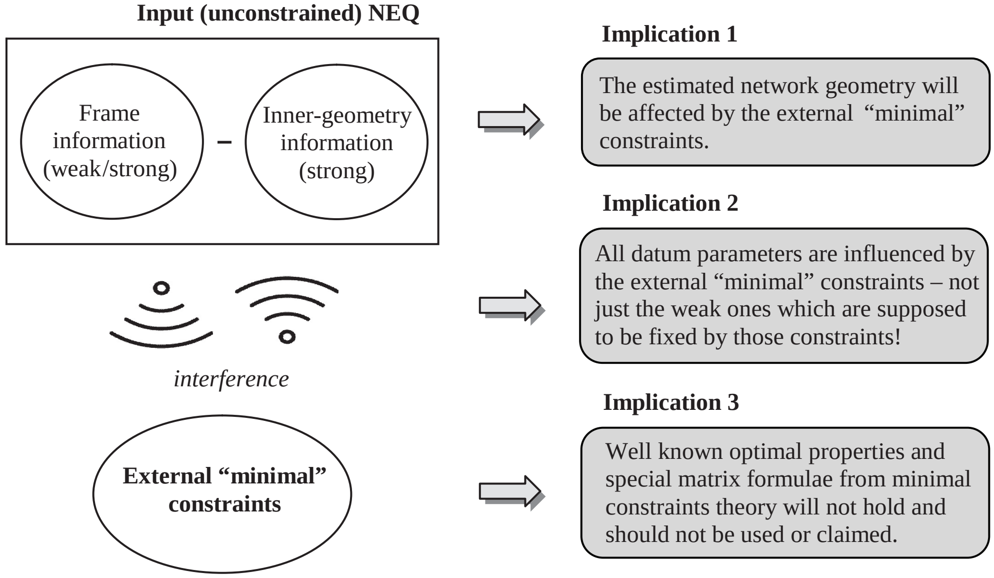


Figure 1. *Theoretical and practical implications due to non-singularity in the input NEQ under the minimal-constraint estimation framework. Note that a large weight matrix for the external constraints may eliminate the impact of the NEQ content on the external datum definition, but it cannot prevent the distortion of the network geometry and the alteration of well-estimable datum parameters which are not included in the external constraints (more explanations are provided in following sections of the paper).*

parameters while retaining the rest of the data information. The input NEQ under consideration are the ones obtained by geodetic data processing in the context of TRF estimation based on the Gauss-Markov linear adjustment model (e.g. Angermann et al. 2004; Bloßfeld 2015). This covers all usual schemes of NEQ-based frame realizations, including (i) epoch solutions by single network analysis or subnetwork combination, (ii) cumulative solutions by multi-epoch stacking, and (iii) inter-technique combination solutions. The non-singularity of the input NEQ in any of the above cases raises concerns which will be addressed in the rest of this study by answering the following questions:

- *how to identify the rank defect and, especially, how to assess the datum information carried by the (unconstrained) NEQ in a geodetic network;*
- *how significant is the distortion caused in a minimally-constrained network solution due to the absence of true rank defect in the input NEQ;*
- *how to convert NEQ to a truly singular system with given rank defect for particular frame components, without altering the NEQ's geometrical content.*

This study investigates the above issues and presents an algebraic approach to enforce the proper (or desired) singularity in NEQ systems for TRF studies. The motivation of our analysis is twofold and it aims to accommodate either the recovery of theoretically expected singularities, or the intentional removal of distrusted datum information that already exists in the available NEQ. Although our examples refer only to cases of Global Navigation Satellite Systems (GNSS) networks with fixed satellite orbits and Earth rotation parameters (ERPs), the rationale and the mathematical formulation of this study are applicable to any type of frame realization or network adjustment problem from geodetic data.

2. Main aspects of the problem

Let us consider a NEQ system as obtained by the analysis of space geodetic observations for estimating the network coordinates $\boldsymbol{X}$ in static or linear-kinematic mode (i.e. the vector $\boldsymbol{X}$ may contain either station positions or station positions+velocities). It is assumed that any nuisance parameters have been reduced or eliminated beforehand, and the input NEQ are given in the usual linearized form:

$$\boldsymbol{N}\left(\boldsymbol{X}-\boldsymbol{X}_o\right) = \boldsymbol{u} \tag{1}$$

where $\boldsymbol{N}$ is the normal matrix, $\boldsymbol{X}_o$ is the vector of the a priori coordinates, and $\boldsymbol{u}$ is the right-hand side normal vector. The formation of such NEQ in practice, either by single network analysis or by subnetwork combination through stacking procedures, may often lead to a full rank system. This is not surprising and it can occur due to datum information which is carried by various modeling choices or software-dependent procedures during the data analysis stage. The existence of this situation and its consequences for geodetic network analysis were first discussed in Davies and Blewitt (2000) and later by Kelm (2003). It has been also implied in NEQ combination strategies as presented by Seitz et al. (2012, see for instance Eqs. (1)-(2)) and in the computation of "free" network solutions via the regular inverse of $\boldsymbol{N}$ (Dach et al. 2015, p. 247). Lastly, it was considered by Rebischung et al. (2016) in the pre-processing of daily SINEX (Solution Independent Exchange Format) files from the International GNSS Service (IGS) Analysis Centers, along with a corrective scheme which will be further discussed in Sect. 4.

To demonstrate the occurrence of the aforementioned situation, we give in Fig. 2 some examples of the eigenvalue spectra for (unconstrained) normal matrices that have been obtained from weekly SINEX files in three different GNSS networks: the European Reference Frame Permanent Network (EPN), the South American Reference Frame Network (SIRGAS), and the Asia-Pacific Reference Frame Network (APREF). In all cases, the NEQ were reconstructed via the standard deconstraining (IERS 2006, appendix II) based on the stated information for the applied constraints in the stored weekly solutions. The smallest eigenvalues of the

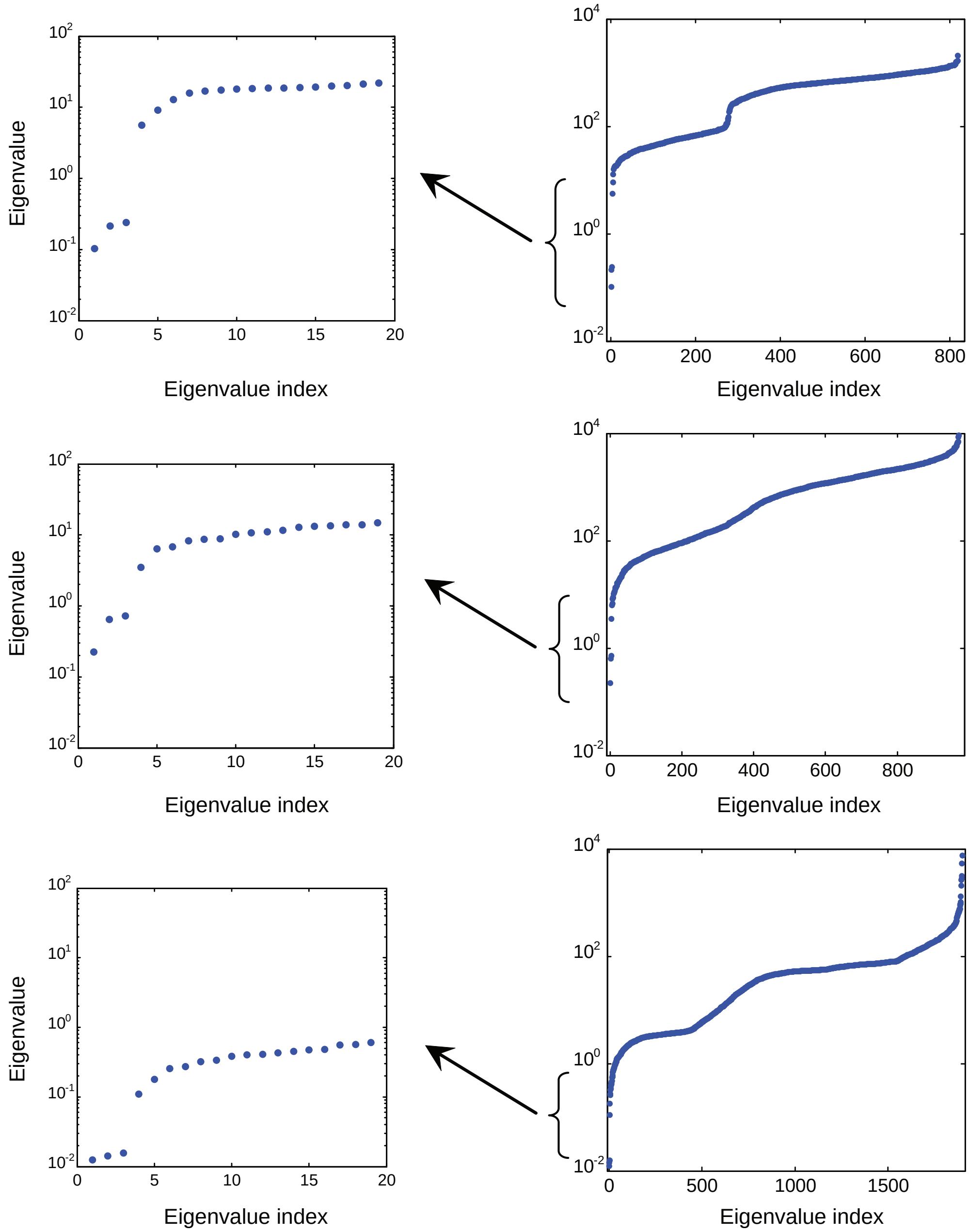


Figure 2. *Eigenvalues of weekly normal matrices for three different GNSS networks: EPN (upper plots), SIRGAS (middle plots) and APREF (bottom plots). The unconstrained NEQ have been reconstructed from the stated information in the corresponding SINEX files: EUR18697.snx, SIR18697.snx, AUS18697.snx.*

recovered normal matrices range from 0.01 to 0.2, and they largely reflect the weak origin information due to the fixation of the satellite orbits in the respective networks. The ratio between the maximum and minimum eigenvalues in these examples is 10^4-10^5 while the apparent gap in the lower band of the spectrum does not exceed one order of magnitude, thus implying a rather well-conditioned invertible normal matrix without any real rank deficiency! Analogous behavior can be found in the daily NEQ of these networks, as well as in their multi-year stacked NEQ where both station positions and velocities appear in the unknown parameters of Eq. (1).

Although the geometry of the estimated polyhedron is well determined by invertible NEQ like the ones considered in the example of Fig. 2, it will still refer to a loose coordinate system which can be far from the desired frame for the underlying network. The datum definition, or at least a part of it, is thus always implemented by external conditions which complement the weak data-provided frame information. Various tools have been developed for this purpose and they are regularly used to handle the datum choice problem in geodetic frame realizations. The so-called minimal constraints (MCs) and the related no-net-translation (NNT), no-net-rotation (NNR), no-net-scaling (NNS) conditions are primary examples of such tools, which support the realization of high-quality TRFs without theoretically interfering with the well-estimable content of the observations; see e.g. Altamimi et al. (2002), Altamimi (2003), Angermann et al. (2004), Seitz et al. (2012) and Glaser et al. (2015). In practice, however, the MC implementation could affect all estimable network characteristics since the addition of datum constraints to a normal matrix that is already invertible (as in Fig. 2) will generally yield a distorted solution. Hence, a reasonable concern for TRF studies is whether the regularity of the input NEQ may cause a sizeable network distortion even when the practically minimal constraints − in the sense described by Sillard and Boucher (2001) − are applied towards the inversion of Eq. (1). An attempt to answer this question is given in Sect. 3.

The scope of this study is to formulate an analytic procedure, hereafter called *controlled datum removal* (*CDR*), for eliminating prior datum information from available NEQ in geodetic networks. Its role is to allow the proper implementation of MC theory in frame realization either for single-epoch or multi-epoch cumulative network solutions. In essence what we aim for is to ensure that a chosen set of datum conditions will be a rightful choice of minimal constraints for TRF estimation problems! To meet this requirement we have to remove beforehand any information related to datum parameters that we intend to define through the external minimal constraints and, at the same time, preserve the original NEQ content about the network geometry (and also about estimable frame components that we do not wish to be handled by the external constraints).

The rationale of our analysis resembles the assessment of inner accuracy in geodetic networks. The latter is a useful concept that was introduced in geodesy by Meissl (1965, 1969) in the context of a filtering process for removing the effect of an arbitrary set of implicit parameters out of a given covariance (CV) matrix. The filtered parameters are the shifts, rotations and scale of a coordinate system, whereas the inner accuracy corresponds to the quality of a coordinate set after having removed the influence of its underlying reference frame. Complementary to this concept is also the covariance-based treatment of Sillard and Boucher (2001) for determining the frame sensitivity of space geodetic techniques in regional or global networks. In contrast to these approaches, our study presents (and advocates the necessity of) a "frame filtering" process that can be applied directly at the NEQ level, without any need to employ intermediate covariance matrices.

As a final remark, let us note that the CDR should not be confused with the usual de-constraining which is often applied in TRF applications. The latter is a standard tool for removing the additive constraints that were used to obtain a network solution based on *stated information* that is stored in SINEX files (e.g. IERS 2006, appendix II). The CDR, on the other hand, is a tool towards the removal of datum information in NEQ which have been already recovered from SINEX files, or formed directly by the analysis of geodetic observations. Its use will guarantee the rank defect with regard to datum parameters that will be handled under the MC setting – more details and the related mathematical formulation will be given in Sect. 4.

3. NEQ diagnostic analysis

3.1 General remarks

There are various ways to infer the datum content of a normal matrix and to check whether it has a true rank defect with regard to one or more frame components (origin, orientation, scale). The analysis of its eigenvalues is a basic tool that can reveal the rank defect in terms of the number of its zero, or close to zero, eigenvalues. Note that a rank-deficient matrix is characterized by a cluster of small eigenvalues with a *sizeable gap* relative to its larger eigenvalues (Hansen 1998, p. 2). If such rank deficiency cannot be numerically justified, as in the cases shown in Fig. 2, the lower part of the eigenvalue spectrum can still expose the presence of ill-defined frame components but without being able to specify their particular type. It is emphasized that NEQ obtained by the same observation technique and modeling procedure over the same network – thus carrying equivalent datum content – may exhibit different variations in their eigenvalue spectrum as shown in the example of Fig. 3.

The type of rank defect is usually deduced from theoretical considerations by taking into account the parametric modeling and spatial extent of the underlying net-

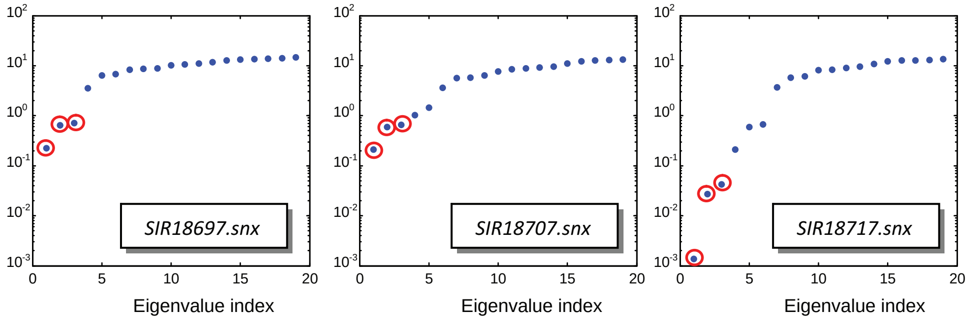


Figure 3. *Eigenvalues of unconstrained normal matrices from different weekly combined solutions of the SIRGAS network (only the first 19 values are shown). The three smallest eigenvalues are indicated in red circles.*

work. For example, the normal system obtained from the processing of GNSS observations in a global network for the simultaneous estimation of satellite orbits, ERPs and stations positions, is expected to have a rank defect equal to three which corresponds to the three degrees of freedom of the unobserved network orientation. On the other hand, the rank defect of GNSS networks with fixed satellite orbits and ERPs is theoretically expected to be zero, yet in practice it also appears to be close to three as a result of the weak origin definition, especially in networks with non-global coverage. In both cases we may still end up with non-singular input NEQ (e.g. Davies and Blewitt 2000) which need to be properly handled in the context of TRF estimation.

The quantitative assessment of frame sensitivity in observed networks by space geodetic techniques relies on the concept of the "reference system effect" which was introduced by Sillard and Boucher (2001). A geometric-like approach can be also used for this purpose by assessing the orthogonality level of the two basic subspaces of the TRF estimation problem: the *data-related subspace* spanned by the rows or columns of the input normal matrix $\boldsymbol{N}$, and the *datum-related subspace* spanned by the rows of the so-called Helmert transformation matrix that is denoted hereafter by $\boldsymbol{G}$. This essentially corresponds to checking the validity of the fundamental equation $\boldsymbol{N}\boldsymbol{G}^T = \boldsymbol{0}$ which holds a key role in MC theory of geodetic networks (Blaha 1971; Kotsakis 2012).

From a user's perspective, it is also crucial to investigate the distortion caused by additive datum constraints (which are supposedly "minimal") when the input NEQ do not have a proper rank defect. By the term distortion here we mean the differences in the well-estimable characteristics of a free-net solution and a "minimally-constrained" solution. These differences reflect the corruption of data-related information in the constrained solution by the external datum conditions. For example, if NNT constraints are used to align a regional GNSS network to ITRF, then both the inner geometry and the orientation/scale of the estimated solution could be affected by those constraints. Presumably, one would suspect that this effect is negligible due to the weakness of frame origin information in the input NEQ, yet to the author's knowledge no actual evaluation of this distorting effect exists in the geodetic literature.

In the next sections we look more closely at the above issues through numerical examples using weekly SINEX files from the EPN, SIRGAS and APREF networks.

3.2 Statistical assessment of frame sensitivity

The datum defect of a normal matrix $\boldsymbol{N}$ can be inferred via the covariance matrix $\boldsymbol{\Sigma} = (\boldsymbol{GNG}^T)^{-1}$ which quantifies the so-called *reference system effect*, as explained by Sillard and Boucher (2001). The ill-defined frame components correspond to the large diagonal elements of $\boldsymbol{\Sigma}$ in the sense that their values appear significantly higher than the usual uncertainty of the geodetic observation technique at hand. This does not imply, though, that the normal matrix will be rank-deficient (i.e. singular) in a strict numerical sense. As an example, the reference system effect of the same NEQ which were previously analyzed in terms of their spectral content (see Fig. 2) is given in Table 1. In all cases the frame origin information is weaker than the frame orientation and scale information, as it is should be normally expected in such GNSS networks.

A similar assessment of the datum defect can be performed via the *weight matrix* $\boldsymbol{Q} = \boldsymbol{GNG}^T$ as suggested by Kelm (2003). In this case the ill-defined frame components correspond to the smaller diagonal elements of $\boldsymbol{Q}$, as indicated in the example of Table 2.

Table 1. *Reference system effect (according to Sillard and Boucher 2001) in weekly NEQ of different GNSS networks. The unconstrained normal matrices were recovered from the SINEX files EUR18697.snx, SIR18697.snx and AUS18697.snx, respectively. The values correspond to the square roots of the diagonal elements of the CV matrix* $\boldsymbol{\Sigma}$*. The conversion to linear units (for the frame orientation/scale uncertainty) uses the Earth's radius value R = 6378137 m.*

	EPN	*SIRGAS*	*APREF*
Origin – *x* translation (cm)	17.7	8.1	24.6
Origin – *y* translation (cm)	15.5	10.9	24.8
Origin – *z* translation (cm)	20.1	5.8	24.3
Orientation – *x* rotation (mas/cm)	1.84/5.7	0.51/1.6	1.63/5.0
Orientation – *y* rotation (mas/cm)	2.36/7.3	0.37/1.1	1.58/4.9
Orientation – *z* rotation (mas/cm)	1.55/4.8	0.61/1.9	1.57/4.9
Scale (ppb/cm)	7.46/4.8	3.94/2.5	6.62/4.2

Table 2. *Reference system effect (according to Kelm 2003) in weekly NEQ of different GNSS networks. The unconstrained normal matrices were recovered from the SINEX files EUR18697.snx, SIR18697.snx and AUS18697.snx, respectively. The values correspond to the diagonal elements of the weight matrix **Q**. Note that the frame orientation/scale weights have been numerically re-scaled by the factor $1/R^2$, where R corresponds to the mean Earth radius R = 6378137 m.*

	EPN	*SIRGAS*	*APREF*
Origin – x translation	113.261	283.619	17.797
Origin – y translation	90.173	383.297	17.396
Origin – z translation	115.842	349.593	17.295
Orientation – x rotation	1761.526	8134.032	401.174
Orientation – y rotation	354.960	15053.847	426.739
Orientation – z rotation	2495.045	3330.823	432.638
Scale	3311.158	9454.131	627.080

3.3 Geometrical assessment of frame sensitivity

From a geometrical viewpoint the NEQ datum defect is related to the fact that the rows or columns of the normal matrix are orthogonal to (some or all of) the rows of the Helmert transformation matrix. The latter can be expressed in the standard form:

$$\boldsymbol{G} = \left[\begin{array}{ccccccc} 1 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 1 \\ \hline 0 & z_1 & -y_1 & \cdots & 0 & z_N & -y_N \\ -z_1 & 0 & x_1 & \cdots & -z_N & 0 & x_N \\ y_1 & -x_1 & 0 & \cdots & y_N & -x_N & 0 \\ \hline x_1 & y_1 & z_1 & \cdots & x_N & y_N & z_N \end{array}\right] = \begin{bmatrix} \boldsymbol{g}_1^T \\ \boldsymbol{g}_2^T \\ \vdots \\ \boldsymbol{g}_7^T \end{bmatrix} \tag{2}$$

where the station coordinates refer to approximate values which are close to the sought network solution. The rows of $\boldsymbol{G}$ correspond to the fundamental datum parameters, namely three translations, three rotations and one scale factor. If some of these rows are orthogonal to all rows (or columns) of the normal matrix, this signifies complete lack of data information for the respective frame components and the obligation to fix them by external constraints during the network adjustment. If the NEQ unknowns contain both station positions and velocities, then the fundamental datum parameters increase to 14 and the matrix $\boldsymbol{G}$ should be augmented by 7 additional rows and $3N$ additional columns in accordance to the linearized expression of the Helmert transformation for time-dependent frames (e.g. Sillard and Boucher 2001; Altamimi et al. 2002). In the following we restrict our attention to the 7-parameter case, yet a similar setting can be followed for the more general 14-parameter case.

In principle, the aforementioned orthogonality check can be carried out via the elements of the test matrix:

$$\boldsymbol{N}\boldsymbol{G}^T = \left[\begin{array}{c:c:c:c} \boldsymbol{n}_1^T\boldsymbol{g}_1 & \boldsymbol{n}_1^T\boldsymbol{g}_2 & \cdots & \boldsymbol{n}_1^T\boldsymbol{g}_7 \\ \boldsymbol{n}_2^T\boldsymbol{g}_1 & \boldsymbol{n}_2^T\boldsymbol{g}_2 & \cdots & \boldsymbol{n}_2^T\boldsymbol{g}_7 \\ \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{n}_m^T\boldsymbol{g}_1 & \boldsymbol{n}_m^T\boldsymbol{g}_2 & \cdots & \boldsymbol{n}_m^T\boldsymbol{g}_7 \end{array}\right] \tag{3}$$

where $\{\boldsymbol{n}_i\}$ $(i=1,...,m;\ m=3N)$ denote the columns of the symmetric normal matrix $\boldsymbol{N}$ and $\{\boldsymbol{g}_i\}$ $(i=1,...,7)$ correspond to the columns of the matrix $\boldsymbol{G}^T$. Note that the elements of the above test matrix are strongly influenced by the significant differences among the Euclidean norms for the seven rows of the Helmert transformation matrix; see Eq. (2). This can obscure the appraisal of the datum content in the input NEQ and it may under-estimate the orthogonality level between the columns of $\boldsymbol{N}$ and $\boldsymbol{G}^T$. To detect the frame sensitivity it is therefore better to rely on the normalized test matrix:

$$\widetilde{\boldsymbol{N}\boldsymbol{G}}^T = \left[\begin{array}{c:c:c:c} \dfrac{\boldsymbol{n}_1^T\boldsymbol{g}_1}{\|\boldsymbol{n}_1\|\cdot\|\boldsymbol{g}_1\|} & \dfrac{\boldsymbol{n}_1^T\boldsymbol{g}_2}{\|\boldsymbol{n}_1\|\cdot\|\boldsymbol{g}_2\|} & \cdots & \dfrac{\boldsymbol{n}_1^T\boldsymbol{g}_7}{\|\boldsymbol{n}_1\|\cdot\|\boldsymbol{g}_7\|} \\ \dfrac{\boldsymbol{n}_2^T\boldsymbol{g}_1}{\|\boldsymbol{n}_2\|\cdot\|\boldsymbol{g}_1\|} & \dfrac{\boldsymbol{n}_2^T\boldsymbol{g}_2}{\|\boldsymbol{n}_2\|\cdot\|\boldsymbol{g}_2\|} & \cdots & \dfrac{\boldsymbol{n}_2^T\boldsymbol{g}_7}{\|\boldsymbol{n}_2\|\cdot\|\boldsymbol{g}_7\|} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\boldsymbol{n}_m^T\boldsymbol{g}_1}{\|\boldsymbol{n}_m\|\cdot\|\boldsymbol{g}_1\|} & \dfrac{\boldsymbol{n}_m^T\boldsymbol{g}_2}{\|\boldsymbol{n}_m\|\cdot\|\boldsymbol{g}_2\|} & \cdots & \dfrac{\boldsymbol{n}_m^T\boldsymbol{g}_7}{\|\boldsymbol{n}_m\|\cdot\|\boldsymbol{g}_7\|} \end{array}\right] \tag{4}$$

whose elements represent the *cosine angles* between the vector sets $\{\boldsymbol{n}_i\}$ and $\{\boldsymbol{g}_i\}$, thus always ranging between -1 and 1. The closer the elements of each column in the above matrix are to zero, the less information is contained in the normal matrix for the corresponding datum parameter. A relevant example is given in Fig. 4 using the weekly (unconstrained) NEQ of different GNSS networks. As expected, the frame origin parameters show higher orthogonality level in relation to the normal matrix than the other datum components.

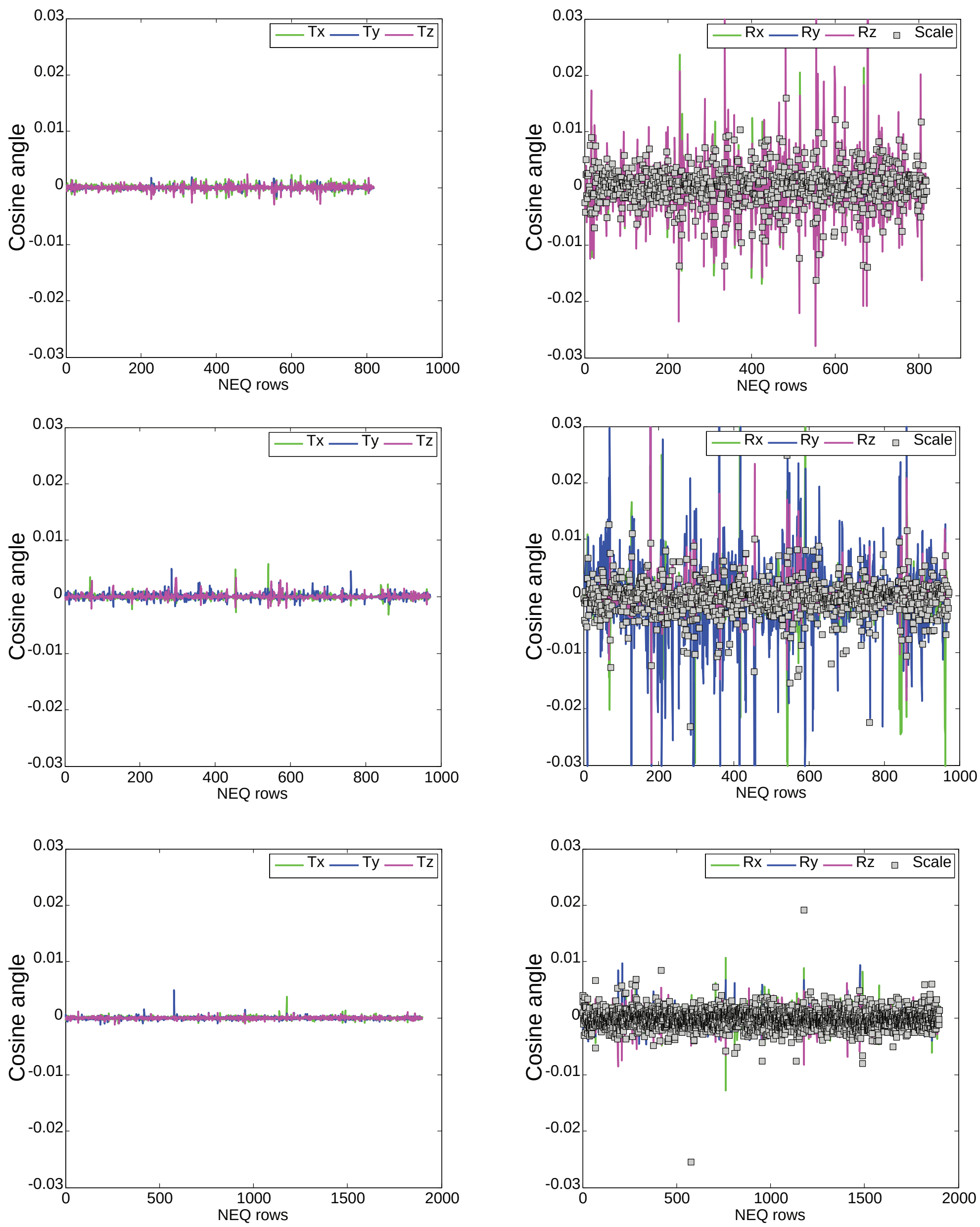
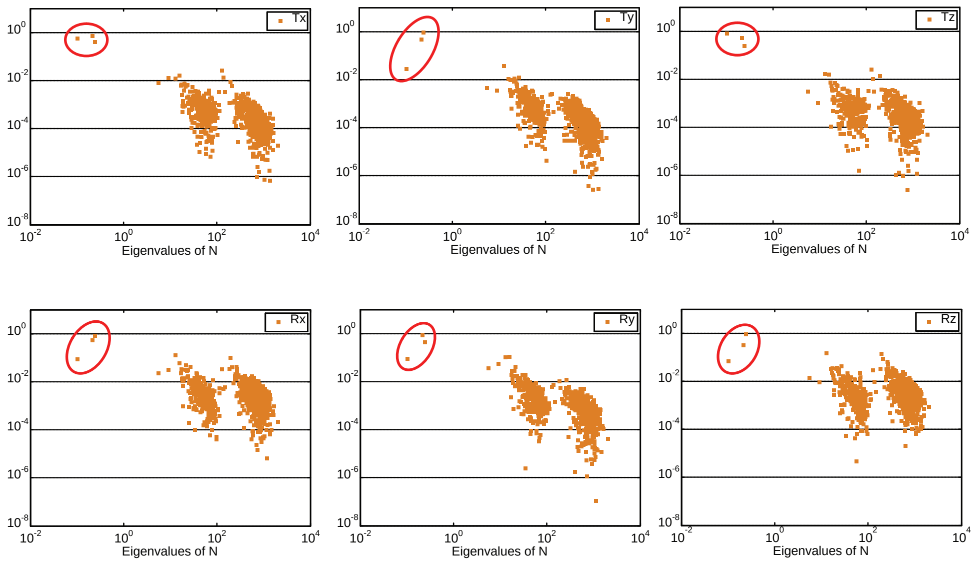


Figure 4. *Cosine angles between the seven rows of the Helmert transformation matrix and the rows of weekly normal matrices in different GNSS networks: EPN (upper plots), SIRGAS (middle plots) and APREF (bottom plots). The unconstrained NEQ were recovered from the corresponding weekly solutions for the time period 1-7/11/2015.*

3.4 *Spectral assessment of frame sensitivity*

The previous "geometrical" assessment can be equivalently applied in the spectral domain by considering the eigenvalue decomposition of the normal matrix, that is $\boldsymbol{N} = \boldsymbol{U}^T \boldsymbol{D} \boldsymbol{U}$. The normalized test matrix $\widetilde{\boldsymbol{NG}}^T$ in Eq. (4) can then be replaced by a similar test matrix $\widetilde{\boldsymbol{UG}}^T$ which employs the eigenvectors $\{\boldsymbol{u}_i\}$ instead of the columns $\{\boldsymbol{n}_i\}$ of the normal matrix. The advantage of this option is that it provides the means to analyze the datum content of a NEQ system as a function of its eigenvalues, and identify the frame components whose information is mostly concentrated on the weak part of the normal matrix spectrum.

In Fig. 5 we show an example of such a spectral assessment in the EPN network using the weekly NEQ that were employed in the previous tests. The cosine angles between the rows of the Helmert transformation matrix and the eigenvectors of the normal matrix are now reduced in the interval [0,1] (i.e. the angles are considered in the range 0°-90° instead of 0°-180°) to allow their visualization in logarithmic



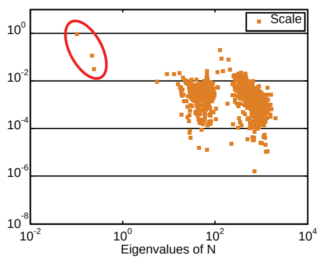


Figure 5. *Cosine angles between the seven rows of the Helmert transformation matrix and the eigenvectors of a weekly normal matrix in the EPN network. Their values are plotted against the associated eigenvalues of the corresponding eigenvectors. The NEQ were recovered from the combined weekly solution for the time period 1-7/11/2015.*

scale. Some important aspects that can be inferred from this example are summarized below.

- The cosine angles between the vector sets $\{\boldsymbol{g}_i\}$ & $\{\boldsymbol{u}_i\}$ have larger variations compared to the vector sets $\{\boldsymbol{g}_i\}$ & $\{\boldsymbol{n}_i\}$ that were shown in Fig. 4. This is expected due to the orthogonality of the eigenvector basis $\{\boldsymbol{u}_i\}$ in the R^{3N} Euclidean space.
- The major portion of frame origin information is contained in the weak part of the normal matrix spectrum, particularly in the eigenvectors corresponding to the three smallest eigenvalues.
- The weak part of the normal matrix spectrum carries information for all datum parameters and not just for the frame origin. This information is fully blended in the corresponding eigenvectors since the respective cosine angles approach the value 1.
- The information for the frame orientation and scale is more powerful than the frame origin in the intermediate and strong parts of the normal matrix spectrum, thus revealing their better estimability from the GNSS observations.

3.5 Assessment of distortion in "minimally-constrained" solutions

A straightforward way to assess the network distortion from the minimally-constrained inversion of Eq. (1), due to absence of true rank defect in the input NEQ, is to compare the following solutions:

(i) the unconstrained (free-net) solution

$$\hat{\boldsymbol{X}}_{UNC} = \boldsymbol{X}_o + \boldsymbol{N}^{-1}\boldsymbol{u} \tag{5}$$

which retains the data information for the network geometry and other frame characteristics that are well estimable by the observational model, and

(ii) the constrained solution from a set of "minimal constraints", $\boldsymbol{H}(\boldsymbol{X}-\boldsymbol{X}_o)=\boldsymbol{c}$, chosen according to the diagnostic analysis of the previous sections, that is

$$\hat{\boldsymbol{X}} = \boldsymbol{X}_o + \left(\boldsymbol{N}+\boldsymbol{H}^T\boldsymbol{W}\boldsymbol{H}\right)^{-1}\left(\boldsymbol{u}+\boldsymbol{H}^T\boldsymbol{W}\boldsymbol{c}\right) \tag{6}$$

where $\boldsymbol{W}$ denotes a datum-related weight matrix. If the input NEQ were truly rank-deficient and the applied constraints fulfill only their datum defect, then the result of Eq. (6) is independent of this weight matrix (e.g. Kotsakis 2012, 2013). For the cases considered here, however, the normal matrix $\boldsymbol{N}$ is non-singular and the choice of $\boldsymbol{W}$ has an impact on the estimated positions. In our following tests this weight matrix is chosen sufficiently large (see below) to outweigh the weak frame infor-

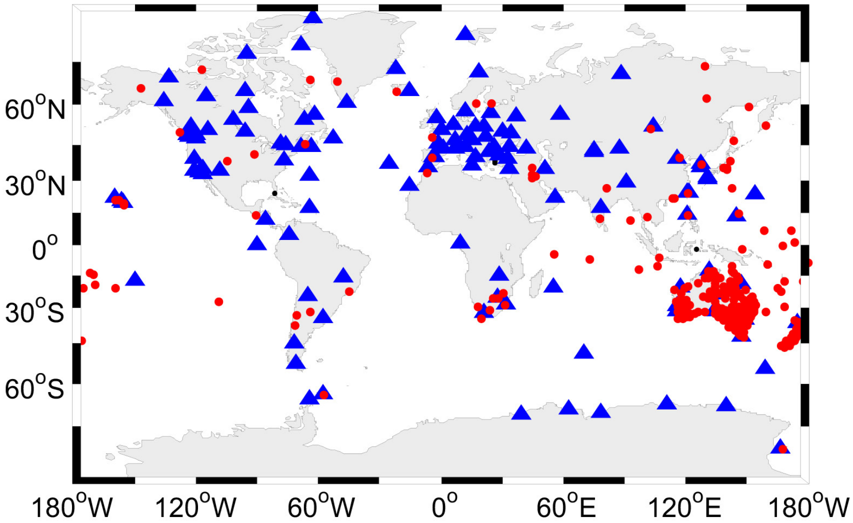


Figure 6. *The APREF GNSS network – the reference stations that were used in the computation of the weekly NNT solution for our numerical test are shown in blue triangles.*

mation in the input NEQ and, thus, to ensure the prevalence of the MC-based datum definition.

The comparison of the above solutions should rely on the position residuals after their fitting by an estimated Helmert transformation over all network stations. Herein we present a relevant example with weekly NEQ from the APREF network. The minimally-constrained solution is computed via the NNT conditions with respect to the IGb08 frame based on the reference stations shown in Fig. 6. The weight matrix $\boldsymbol{W}$ is diagonal and it is tuned to an accuracy level of 10^{-5} m for the frame origin fixation. Three different types of transformation models were applied for fitting the solutions from Eqs. (5) and (6), namely a shift-only model, a shift/rotation model and a full similarity model. The estimated transformation parameters from each model are given in Table 3, whereas the corresponding residuals are plotted separately for each spatial component as shown in Fig. 7.

Our results reveal that the NNT solution is distorted by several mm compared to the free-net solution. Most of this unwanted effect occurs in the vertical component with residuals up to 3-4 mm, whereas the horizontal residuals are approximately two times smaller. Due to the global extent of the APREF network, a uniform scale factor is not able to account for these spatial distortions and the post-fit residuals remain practically the same after the implementation of all types of transformation models (see Fig. 7). The results from analogous comparisons between more weekly (or daily) solutions in the same network demonstrated even larger distortions than the ones presented herein.

Table 3. *Helmert transformation parameters between the free-net and NNT weekly solution in the APREF network. The input NEQ refer to the time period 1-7/11/2015 and they were extracted from the corresponding weekly SINEX file after standard de-constraining.*

	Tx (mm)	Ty (mm)	Tz (mm)	Rx (mas)	Ry (mas)	Rz (mas)	Scale (ppb)
Shift-only	3.2	10.6	7.4	–	–	–	–
Shift/rotation	3.2	10.6	7.5	0.007	-0.000	0.001	–
Full similarity	3.1	10.7	7.4	0.007	-0.000	0.001	-0.05

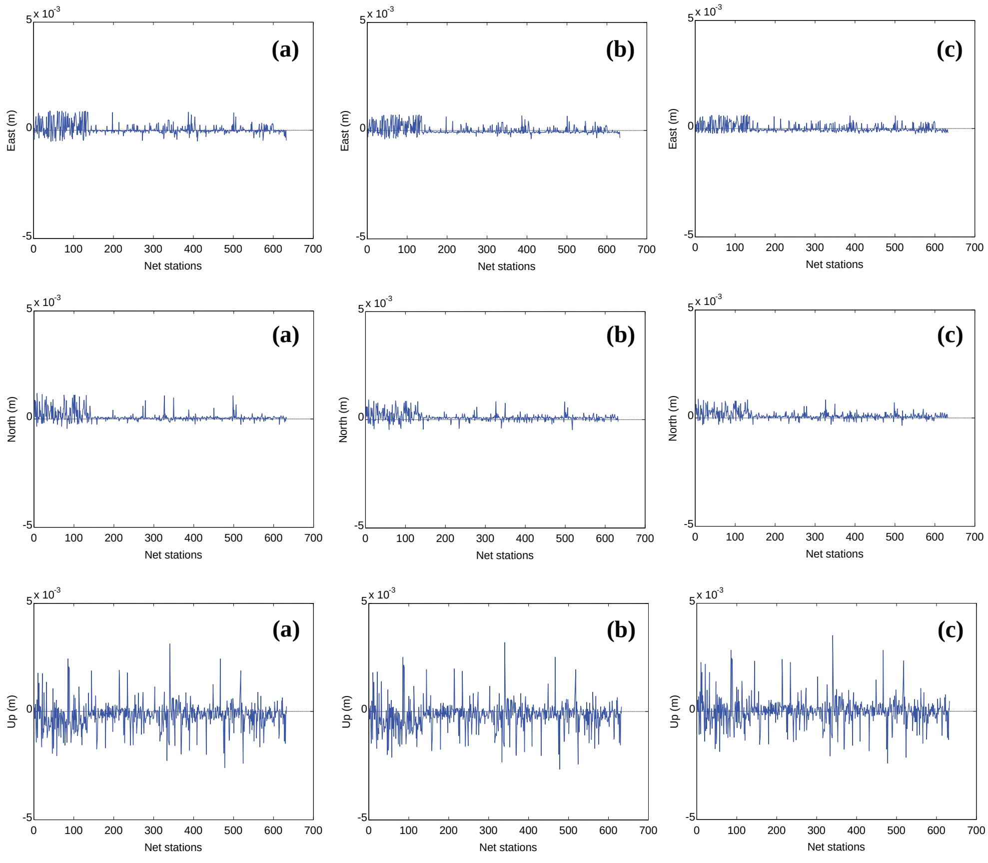


Fig. 7 *Post-fit residuals between the free-net and NNT weekly solutions in the APREF network. The results correspond to three different transformation models: (a) shift-only model, (b) shift/rotation model and (c) full similarity model. The input NEQ refer to the time period 1-7/11/2015.*

It is emphasized that the aforementioned distortions do not imply an incorrect result for the NNT-based estimated positions. They just signify the twisting of the data information towards any subsequent analysis of the adjusted network, and the fact that we do not get a genuine minimally-constrained solution.

4. Removal of frame information from input NEQ

4.1 Mathematical formulation

In this section we consider the problem of conditional conversion of a full-rank normal system to an "equivalent" rank-deficient form which involves the same unknown parameters, that is

$$\boldsymbol{N}\left(\boldsymbol{X}-\boldsymbol{X}_o\right) = \boldsymbol{u} \quad \longrightarrow \quad \boldsymbol{N}'\left(\boldsymbol{X}-\boldsymbol{X}_o\right) = \boldsymbol{u}' \tag{7}$$

For our purpose the initial system ($\boldsymbol{N}$, $\boldsymbol{u}$) refers to the NEQ obtained by the processing of space geodetic data during a least-squares network adjustment or, alternatively, to the NEQ extracted from a SINEX file after removing the additive datum constraints based on stated information for the stored solution. The above conversion is vital in the context of TRF estimation under the MC framework since the aforementioned NEQ do not always exhibit the required datum singularity. In such cases it is essential to be able to reconstruct a singular normal system ($\boldsymbol{N}'$, $\boldsymbol{u}'$) which conforms to particular properties as described below.

Filtering of selected datum parameters. The sought singularity must be attributed to the removal of (weak) datum information that resides in the initial normal matrix. Hence, the following orthogonality condition needs to be fulfilled:

$$\boldsymbol{N}'\boldsymbol{E}^T = \boldsymbol{0} \tag{8}$$

where $\boldsymbol{E}$ is formed by selected rows of the Helmert transformation matrix $\boldsymbol{G}$ from Eq. (2) in the underlying network. These rows correspond to ill-defined datum parameters which we intend to define anew via minimal constraints either in a single network adjustment or in a NEQ stacking/combination procedure.

The fulfillment of Eq. (8) implies the singularity of $\boldsymbol{N}'$ and the fact that its rank defect will be at least equal to the number of rows of $\boldsymbol{E}$ (Blaha 1971, p. 6). In order to ensure that no additional rank defect will exist in the reconstructed NEQ, the new normal matrix should be obtained by a suitable projection of the initial normal matrix as follows

$$\boldsymbol{N}' = \left(\boldsymbol{I}-\boldsymbol{N}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-1}\boldsymbol{E}\right)\boldsymbol{N} \tag{9}$$

(a proof of this equation is given in the appendix). Obviously, this matrix complies with the condition in Eq. (8) and it leads to a singular normal system whose solutions belong in the orthogonal complement of the column space $\Re(\boldsymbol{E}^T)$ using as metric the original matrix $\boldsymbol{N}$ itself.

The reconstructed normal matrix from Eq. (9) is equal to the pseudoinverse (Moore-Penrose inverse) of the singular covariance matrix

$$\boldsymbol{\Sigma_X} \;=\; \left(\boldsymbol{I}-\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{E}^T)^{-1}\boldsymbol{E}\right)\boldsymbol{N}^{-1}\left(\boldsymbol{I}-\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{E}^T)^{-1}\boldsymbol{E}\right) \tag{10}$$

which quantifies the "inner accuracy" of the unconstrained solution from Eq. (5), after having removed the influence of the datum components associated with the rows of $\boldsymbol{E}$. The proof of $\boldsymbol{N}'=\boldsymbol{\Sigma}_{\boldsymbol{X}}^{+}$ is straightforward via simple algebraic operations that can confirm the validity of the four basic properties of the pseudoinverse; see Koch (1999, p. 53).

Upholding the input NEQ information. The initial system $\boldsymbol{N}\left(\boldsymbol{X}-\boldsymbol{X}_o\right)=\boldsymbol{u}$ carries the data information for the network geometry and perhaps for elements of the target frame that are well-estimable by the observational model and do not need to be externally defined by additional constraints. In order to preserve this information, the unconstrained solution from Eq. (5) should satisfy the reconstructed NEQ in the sense that

$$\boldsymbol{N}'\underbrace{\left(\hat{\boldsymbol{X}}_{UNC}-\boldsymbol{X}_o\right)}_{\boldsymbol{N}^{-1}\boldsymbol{u}} \;=\; \boldsymbol{u}' \tag{11}$$

By taking into account Eq. (9), the above condition leads to the projection formula:

$$\boldsymbol{u}' = \left(\boldsymbol{I}-\boldsymbol{N}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-1}\boldsymbol{E}\right)\boldsymbol{u} \tag{12}$$

which defines the constant vector of the reconstructed NEQ in compliance with the filtered datum parameters.

General solution of the reconstructed NEQ. It is deduced without difficulty that any solution of the reconstructed NEQ satisfies the general transformation equation:

$$\hat{\boldsymbol{X}}' \;=\; \hat{\boldsymbol{X}}_{UNC} \;+\; \boldsymbol{E}^T\boldsymbol{\theta} \tag{13}$$

where the (unknown) parameter vector $\boldsymbol{\theta}$ corresponds to the filtered datum components. The last equation represents the well known *S-transformation* within the solution space of $\boldsymbol{N}'\left(\boldsymbol{X}-\boldsymbol{X}_o\right)=\boldsymbol{u}'$, and it reflects the information invariance between the initial and the reconstructed normal systems. The solution $\hat{\boldsymbol{X}}'$ is computable by a suitable set of minimal constraints that can be chosen to overcome the rank defect of $\boldsymbol{N}'$, and it will differ from the unconstrained solution of the initial system only in terms of the datum-related parameters causing the singularity of the reconstructed NEQ. An overview of the preceding CDR framework is given in Table 4.

Connection with other frame filtering approaches. Some aspects of the previous procedure have appeared in the geodetic literature under different viewpoints, all of which are more or less complementary to our frame filtering methodology. Specifically the projection matrix that is used in NEQ reconstruction according to Eqs. (9) and (12), i.e.

$$\boldsymbol{F} = \boldsymbol{I} - \boldsymbol{N}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-1}\boldsymbol{E} \tag{14}$$

is exactly the same projector with the one introduced by Sillard and Boucher (2001) for assessing the reference system effect in geodetic networks. Their algebraic approach is applied entirely in the stochastic domain by decomposing the CV matrix of a (non-constrained) solution to an internal noise component and a datum-related component. Its main objective is to infer the frame sensitivity of space geodetic techniques and it can be used as a *diagnostic tool* for datum defect analysis in NEQ systems, as it was already explained in Sect. 3.2.

In a different context, Rebischung (2014, pp. 186-187) derived Eq. (9) as a *recovery tool* to build the input normal matrix from the inverse CV matrix of a minimally-constrained solution in the absence of explicit knowledge for the datum constraints. This is relevant for cases of analysis centers (ACs) which do not provide the normal matrix of the applied constraints for their reported solutions in SINEX format (i.e. SOLUTION/MATRIX_APRIORI block), thus making impossible the reconstruction of the original NEQ by the standard de-constraining. Indeed, Eq. (9) may be used for this purpose under the assumption that the non-reported constraints correspond to minimal constraints, and by setting the matrix $\boldsymbol{E}$ to be compliant with the theoretical singularities of the input normal matrix. Such an approach was exploited by Rebischung et al. (2016) in the pre-processing of daily solutions from the IGS ACs to enforce the proper rank defect towards their combination for the IGS contribution to the ITRF2014 (Altamimi et al. 2016). This correction step is necessary since the (unconstrained) normal matrices from several ACs do not show the three expected orientation singularities, thus causing a risk of unwanted deformations in the combined IGS solution for the station positions and Earth rotation parameters. It is noted, though, that to avoid any distortion in the TRF estimation process the corresponding (daily) normal vectors should also be properly reconstructed according to Eq. (12); see also Table 4.

Lastly, it is worth mentioning that the CDR methodology of the present study is completely equivalent to the "NEQ augmentation" methodology which was described in Bloßfeld (2015, p. 31). According to his approach, the removal of frame information from a normal system can be implemented by introducing infinitesimal similarity transformation parameters which inflict the expected datum defect in the TRF estimation process. An example for the implementation of this approach can be found in Bloßfeld et al. (2016), and its formal equivalency to our NEQ reconstruction algorithm is demonstrated in the appendix.

Table 4. *Comparison between the initial (full-rank) and the reconstructed (singular) NEQ according to the CDR filtering procedure.*

	Before CDR	After CDR
System of normal equations	$\boldsymbol{N}\left(\boldsymbol{X}-\boldsymbol{X}_o\right)=\boldsymbol{u}$ Initial NEQ without rank defect	$\boldsymbol{N}'\left(\boldsymbol{X}-\boldsymbol{X}_o\right)=\boldsymbol{u}'$ Reconstructed NEQ with "proper" rank defect
Helmert transformation matrix and handling of datum parameters	$\boldsymbol{G}=\begin{bmatrix}\boldsymbol{E}\\ \tilde{\boldsymbol{E}}\end{bmatrix}\begin{matrix}\}\,\boldsymbol{\theta}\\ \}\,\tilde{\boldsymbol{\theta}}\end{matrix}$	$\boldsymbol{\theta}$: ill-defined datum parameters to be fully removed from the NEQ system $\tilde{\boldsymbol{\theta}}$: well-estimable datum parameters from the network data
Evidence of frame information in the NEQ system	$\boldsymbol{N}\boldsymbol{E}^T\neq\boldsymbol{0}$ $\boldsymbol{N}\tilde{\boldsymbol{E}}^T\neq\boldsymbol{0}$	$\boldsymbol{N}'\boldsymbol{E}^T=\boldsymbol{0}$ $\boldsymbol{N}'\tilde{\boldsymbol{E}}^T\neq\boldsymbol{0}$
Filtering of selected datum parameters	$\boldsymbol{N}'=(\boldsymbol{I}-\boldsymbol{N}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-1}\boldsymbol{E})\boldsymbol{N}$ $\boldsymbol{u}'=(\boldsymbol{I}-\boldsymbol{N}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-1}\boldsymbol{E})\boldsymbol{u}$	
NEQ solution	Pseudo minimally constrained $\hat{\boldsymbol{X}}=\boldsymbol{X}_o+(\boldsymbol{N}+\boldsymbol{E}^T\boldsymbol{E})^{-1}\boldsymbol{u}$	Minimally constrained $\hat{\boldsymbol{X}}'=\boldsymbol{X}_o+(\boldsymbol{N}'+\boldsymbol{E}^T\boldsymbol{E})^{-1}\boldsymbol{u}'$
	Note: instead of $\boldsymbol{E}$, *any other MC matrix for fixing the datum parameters* $\boldsymbol{\theta}$ *can be used in the above estimators*	
	Unconstrained $\hat{\boldsymbol{X}}_{UNC}=\boldsymbol{X}_o+\boldsymbol{N}^{-1}\boldsymbol{u}$	S-transformation $\hat{\boldsymbol{X}}'=\hat{\boldsymbol{X}}_{UNC}+\boldsymbol{E}^T\boldsymbol{\theta}$
Other properties	"Inner network accuracy" (to be preserved by CDR) $\boldsymbol{\Sigma}_{\boldsymbol{X}}=(\boldsymbol{I}-\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{E}^T)^{-1}\boldsymbol{E})\times$ $\times\boldsymbol{N}^{-1}(\boldsymbol{I}-\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{E}^T)^{-1}\boldsymbol{E})$	Equivalent computation of the reconstructed NEQ $\boldsymbol{N}'=\boldsymbol{\Sigma}_{\boldsymbol{X}}^{+}$ $\boldsymbol{u}'=\boldsymbol{N}'\left(\hat{\boldsymbol{X}}_{UNC}-\boldsymbol{X}_o\right)$

4.2 Extended use of CDR

Thus far the rationale of this study has relied on two basic presumptions: (a) the *non-singularity* of the input normal matrix $\boldsymbol{N}$, and (b) the need to remove the *ill-defined* datum information from the initial system $\boldsymbol{N}(\boldsymbol{X}-\boldsymbol{X}_o)=\boldsymbol{u}$. Herein we briefly discuss the relaxation of both of these presumptions so that more flexibility is added to the CDR tool for handling TRF estimation problems.

The first generalization is related to the filtering of any subset of datum parameters, regardless how well these parameters are resolved within the input NEQ. This extension is directly applicable as it does not require any modification of the fundamental equations (9) and (12). The matrix $\boldsymbol{E}$ that is used in the CDR algorithm is not actually restricted by any theoretical conditions – other than having linearly independent rows – and it is free to refer to any estimable quantities that have an implicit dependence on the primary parameters $\boldsymbol{X}$ of the normal system. In fact, we may even select this matrix to coincide with the full Helmert transformation matrix ($\boldsymbol{E}=\boldsymbol{G}$) in which case the input NEQ will be completely de-stripped of their datum content without losing their geometrical information. This option can be quite helpful in TRF estimation via intra/inter-technique NEQ combination (e.g. Seitz et al. 2012; Glaser et al. 2015), as it allows to filter out beforehand the datum content of the input data which we do not want to affect the final combination solution.

The second generalization refers to cases where the normal matrix $\boldsymbol{N}$ is already singular and its rank defect is related to a subset of datum parameters, say $\boldsymbol{\theta}_1$. The CDR filtering can still be used to remove the information for another subset of datum parameters, say $\boldsymbol{\theta}_2$, according to the modified expressions:

$$\boldsymbol{N}' = \left(\boldsymbol{I}-\boldsymbol{N}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-}\boldsymbol{E}\right)\boldsymbol{N} \tag{15}$$

and

$$\boldsymbol{u}' = \left(\boldsymbol{I}-\boldsymbol{N}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-}\boldsymbol{E}\right)\boldsymbol{u} \tag{16}$$

where the rows of the matrix $\boldsymbol{E}$ will correspond to the datum parameters of the second subset. The new element here is the presence of the generalized inverse $(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-}$ instead of the regular inverse $(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-1}$. The latter does not necessarily exist since the input normal matrix is supposed to be singular.

The projectors in Eqs. (15)-(16) lead to a new normal system with larger rank defect which is related to both subsets of datum parameters. The orthogonality condition with respect to $\boldsymbol{\theta}_2$, that is $\boldsymbol{N}'\boldsymbol{E}^T=\boldsymbol{0}$, is verified in view of the fundamental property $\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-}(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)=\boldsymbol{E}^T$ (e.g. Koch 1999, p. 51) while the orthogonality to $\boldsymbol{\theta}_1$ is guaranteed by the fact that the normal matrix $\boldsymbol{N}$ is already orthogonal to these datum parameters. Note that both $\boldsymbol{N}'$ and $\boldsymbol{u}'$ remain invariant to the actual choice of generalized inverse in Eqs. (15)-(16) due to the well known invariance of the symmetric matrix $\boldsymbol{E}^T(\boldsymbol{ENE}^T)^-\boldsymbol{E}$.

Moreover, in line with Eq. (11), the preservation of data information in the reconstructed NEQ is ensured by the condition

$$\boldsymbol{N}'\underbrace{\left(\hat{\boldsymbol{X}}^* - \boldsymbol{X}_o\right)}_{\boldsymbol{N}^-\boldsymbol{u}} = \boldsymbol{u}' \tag{17}$$

where $\hat{\boldsymbol{X}}^*$ is an arbitrary solution of the (singular) normal system $\boldsymbol{N}(\boldsymbol{X} - \boldsymbol{X}_o) = \boldsymbol{u}$. Indeed, if we substitute $\boldsymbol{N}'$ and $\boldsymbol{u}'$ from Eqs. (15)-(16) into Eq. (17), then we get

$$\left(\boldsymbol{I} - \boldsymbol{NE}^T(\boldsymbol{ENE}^T)^-\boldsymbol{E}\right)\boldsymbol{N}\boldsymbol{N}^-\boldsymbol{u} = \left(\boldsymbol{I} - \boldsymbol{NE}^T(\boldsymbol{ENE}^T)^-\boldsymbol{E}\right)\boldsymbol{u} \tag{18}$$

which holds true considering the reproducing property $\boldsymbol{N}\boldsymbol{N}^-\boldsymbol{u} = \boldsymbol{u}$ from the generalized inversion theory of normal systems (e.g. Kotsakis 2012, appendix).

5. Summary – Conclusions

The rank defect of the input NEQ is a crucial element for the proper handling of TRF estimation problems under the minimal-constraint framework. This required singularity does not always exist in the unconstrained NEQ which are recovered from SINEX files using the stated information for the stored solutions. The same setback also occurs with the NEQ formed during the least-squares processing of space geodetic data due to the datum information that is often carried by various modeling choices or software-dependent procedures (Davies and Blewitt 2000, Kelm 2003). To verify the presence of this problem, and also to facilitate the assessment of the datum content of normal systems in geodetic networks, a number of (algebraic, statistical, geometrical and spectral) diagnostic tools have been studied in this paper; see Sect. 3.

To avoid distortion effects in minimally-constrained networks, the input NEQ should be pre-filtered by stripping them of their datum content, yet without affecting their geometrical information, according to the CDR algorithm given in Sect. 4. This approach serves essentially a similar purpose as the *loosening transformation* (Blewitt 1998) or the *Helmert-based stacking model* (Altamimi et al. 2002), both of which have been used in the context of TRF estimation for the optimal combination of individual solutions from different sub-networks, measurement epochs and observation techniques. In contrast to these tools which operate at the "solution level", the CDR filtering is applied at the "NEQ level" and guarantees the rank defect with regard to datum parameters that will be handled under the MC setting, without the need to employ intermediate covariance matrices or other auxiliary transformation parameters. Note that, even in combination schemes at the solution level, the CDR is still a desirable and useful tool that can ensure the availability of truly distortion-free solutions (from corresponding filtered NEQs) to be assimilated in the TRF estimation process.

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Appendix

The equivalency between the CDR filtering that was presented in Sect. 4.1 and the related methodology for deriving datum-free NEQ according to Bloßfeld (2015, p. 31) will be demonstrated here.

Let us consider a normal system $\boldsymbol{N}\left(\boldsymbol{X}-\boldsymbol{X}_o\right)=\boldsymbol{u}$ that is obtained from the analysis of space geodetic observations for estimating the station positions $\boldsymbol{X}$ in a geodetic network with respect to a target frame. It is assumed that this system has full rank due to datum information which is present in the form of (unknown) minimal constraints. This setting can cover two different cases of "input NEQ" with practical relevance for TRF estimation problems, namely:

(i) minimally-constrained NEQ recovered from SINEX files which do not report the information for the applied constraints in the stored solutions; or

(ii) de-constrained NEQ recovered from SINEX files which report the information for the applied constraints in the stored solutions.

The former case is considered in Bloßfeld (2015) and Rebischung (2014), yet the latter is also of interest as it could require the implementation of an additional datum removal scheme (see the discussion and examples in Sect. 2).

The invertible normal system $\boldsymbol{N}\left(\boldsymbol{X}-\boldsymbol{X}_o\right)=\boldsymbol{u}$ can be algebraically associated with a "fictitious" system of observation equations, in the sense that

$$\boldsymbol{N}=\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}\ ,\quad \boldsymbol{u}=\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{b} \tag{A1}$$

where $\boldsymbol{A}$, $\boldsymbol{P}$ and $\boldsymbol{b}$ stem from a full-rank linear Gauss-Markov model

$$\boldsymbol{b}=\boldsymbol{A}\left(\boldsymbol{X}-\boldsymbol{X}_o\right)+\boldsymbol{v}\ ,\qquad \boldsymbol{v}\sim(\boldsymbol{0},\ \sigma^2\boldsymbol{P}^{-1}) \tag{A2}$$

The selective removal of datum information from the normal system can be implemented by introducing an artificial frame-related rank defect in the above system of observation equations. This is achieved via a simple re-parameterization using the Helmert transformation model

$$\boldsymbol{X}=\boldsymbol{X}'+\boldsymbol{E}^T\boldsymbol{\theta} \tag{A3}$$

where the elements of $\boldsymbol{\theta}$ (and the rows of the transformation matrix $\boldsymbol{E}$) correspond to the datum parameters that we wish to filter out. By substituting Eq. (A3) into Eq. (A2) we obtain the extended system of observation equations

$$\boldsymbol{b}=\boldsymbol{A}\left(\boldsymbol{X}'-\boldsymbol{X}_o\right)+\boldsymbol{A}\boldsymbol{E}^T\boldsymbol{\theta}+\boldsymbol{v}\ ,\qquad \boldsymbol{v}\sim(\boldsymbol{0},\ \sigma^2\boldsymbol{P}^{-1}) \tag{A4}$$

which, in turn, is linked with the augmented normal system

$$\begin{bmatrix}\boldsymbol{N} & \boldsymbol{N}\boldsymbol{E}^T\\ \boldsymbol{E}\boldsymbol{N} & \boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T\end{bmatrix}\begin{bmatrix}\boldsymbol{X}'-\boldsymbol{X}_o\\ \boldsymbol{\theta}\end{bmatrix}=\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{E}\boldsymbol{u}\end{bmatrix} \tag{A5}$$

Obviously the above system retains the same information about the network's geometrical characteristics as the original system $\boldsymbol{N}\left(\boldsymbol{X}-\boldsymbol{X}_o\right)=\boldsymbol{u}$, but it is singular since the parameter vectors $\boldsymbol{X}'$ and $\boldsymbol{\theta}$ cannot be separately estimated from the same observations.

If we reduce the (unknown) datum parameters from Eq. (A5), then the following "filtered" normal system is derived

$$\left(\boldsymbol{N}-\boldsymbol{N}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-1}\boldsymbol{E}\boldsymbol{N}\right)\left(\boldsymbol{X}'-\boldsymbol{X}_o\right) = \boldsymbol{u}-\boldsymbol{N}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-1}\boldsymbol{E}\boldsymbol{u} \tag{A6}$$

or, equivalently

$$\left(\boldsymbol{I}-\boldsymbol{N}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-1}\boldsymbol{E}\right)\boldsymbol{N}\left(\boldsymbol{X}'-\boldsymbol{X}_o\right) = \left(\boldsymbol{I}-\boldsymbol{N}\boldsymbol{E}^T(\boldsymbol{E}\boldsymbol{N}\boldsymbol{E}^T)^{-1}\boldsymbol{E}\right)\boldsymbol{u} \tag{A7}$$

The last equation is identical to the reconstructed singular NEQ according to Eqs. (9) and (12) of the present paper.